Chapter 1

Continuous time dynamical systems: a quick journey

In this course we saw so far only examples and dynamical properties of *discrete time* dynamical systems. In this section, we briefly mention some of the corresponding definitions for *continuous time* dynamical systems to show that the theory is very similar. Furthermore, in many situations, the study of a continuous time dynamical system can be reduced to the study of a discrete time dynamical system by considering the *first return* or *Poincaré map* on a section. We will present here two examples of continuous dynamical systems, the linear flow on a torus and the billiard in a circle, that, by this procedure, can be reduced to a rotation on the circle.

1.1 Flows: definition, examples and ergodic properties

The evolution of a continuous dynamical system is described by $1-$parameter family of maps $f_t : X \to X$ where $t \in \mathbb{R}$ should be thought as *continuous time parameter* and $f_t(x)$ is the position of the point x after time t. The family of maps $\{f_t\}_{t\in\mathbb{R}}$ should satisfy the properties of a flow:

Definition 1.1.1. *A* 1*-parameter family* $\{f_t\}_{t\in\mathbb{R}}$ *is called a* flow *if* f_0 *is the identity map and for all* $t, s \in \mathbb{R}$ *we have* $f_{t+s} = f_t \circ f_s$*, i.e.*

$$f_{t+s}(x) = f_t(f_s(x)) = f_s(f_t(x)), \quad \textit{for all } x \in X.$$

A continuous time dynamical system $f_t : X \to X$ *is a space* X *together with a* flow $\{f_t\}_{t\in\mathbb{R}}$*. The orbit* $\mathcal{O}^+_{f_t}(x) = \{f_t(x), t \geq 0\}$ *of the flow is also called the* trajectory *of the point* x *under the flow.*

The equation of a flow simply says that the position of the point x after time $t+s$ can be obtained flowing for time s from the point $f_t(x)$ reached by x after time t. Remark that each f_t, $t \in \mathbb{R}$, is *invertible* since it follows from the definition of flow that $f_{-t} = f_t^{-1}$.

Let us give some examples of flows.

Example 1.1.1. [**Linear Flow on** $\mathbb{T}^2$] Let $\mathbb{T}^2 = [0,1]^2/\sim$ be the two-dimensional torus, that is the unit square with opposite sides identified by the relation $(x,0) \sim (x,1)$ and $(0,y) \sim (1,y)$ (which glues top and bottom and right and left sides respectively). Let $\theta \in [0, \pi/2)$ be an angle.

The *linear flow* $\varphi_t^\theta : \mathbb{T}^2 \to \mathbb{T}^2$ in direction θ is the flow $\varphi_t^\theta : \mathbb{T}^2 \to \mathbb{T}^2$ is given by

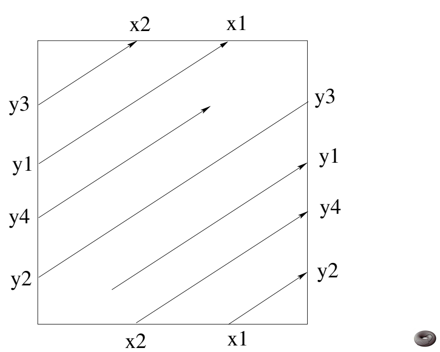
$$\varphi_t^\theta(x, y) = (x(t), y(t)), \qquad \text{where} \qquad \begin{cases} x(t) = x + t\cos\theta \mod 1, \\ y(t) = y + t\sin\theta \mod 1, \end{cases}$$

are solutions of the differential equations

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = \cos\theta \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \sin\theta. \end{cases}$$

The point $\varphi_t^\theta(x, y)$ is obtained by considering the point $(x + t\cos\theta, y + t\sin\theta) \in \mathbb{R}^2$ which belongs to the line through (x, y) in direction θ and has distance t from the initial point (x, y) and projecting it on $\mathbb{T}^2$ by $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ given by $\pi(x, y) = (x \mod 1, y \mod 1)$.

In other words, $\varphi_t^\theta(x, y)$, as $t \geq 0$ increases, moves along the line trough (x, y) in direction θ until $\varphi_t^\theta(x, y)$ hits the boundary of the unit square. After the trajectory hits the boundary of the square, it continuous on the opposite identified side of the square (see Figure 1.1(a)). If we draw the trajectories on the surface of a doughnut (which is another way of representating $\mathbb{T}^2$), the trajectories are continuous and look as in Figure 1.1(b).



(a) (b)

Figure 1.1: A trajectory of the linear flow φ_t^θ on the square (a) and on the doughnut (b).

Example 1.1.2. [**Planar billiards**] Let $D \subset \mathbb{R}^2$ be a closed subset of $\mathbb{R}^2$, for example a rectangle, a polygon or a circle, or more in general, any D whose boundary is a smooth curve with finitely many corners, see Figure 1.2.

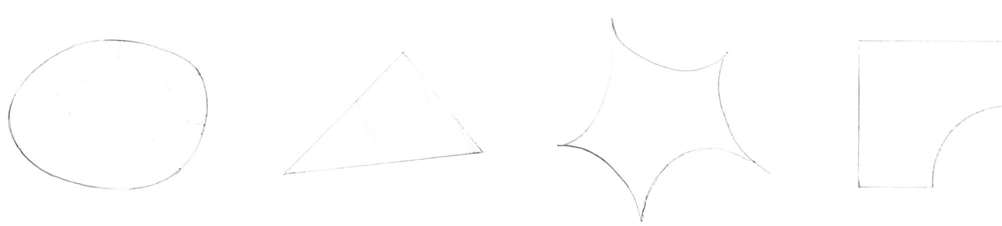


Figure 1.2: Billiard tables.

Consider a point $x \in D$ and a direction $\theta \in [0, 2\pi)$. Imagine shooting a billiard ball from x in direction θ on an ideal billiard table which is frictionless. The ball will move with unit speed along a billiard trajectory which is a straight line in direction θ until it hits the boundary of D. At the boundary, if there is no friction, there is an elastic collision, that is the trajectory is reflected according to the *law of optics* (an exampe is shown in Figure **??**):

Law of optics*: the* angle of incidence*, that is the angle between the trajectory before the collision and the tangent line to the boundary, is* equal *to the* angle of reflection*, that is the angle between the trajectory after the collision and the tangent line to the boundary.*

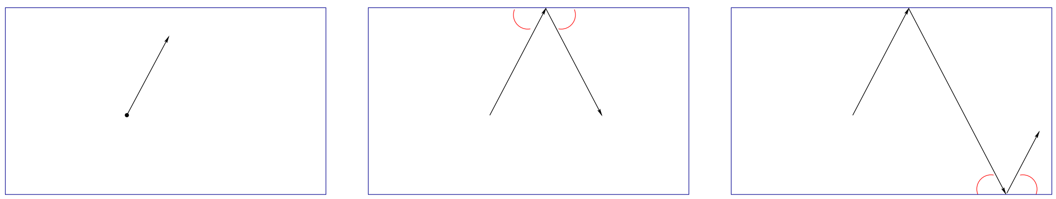


Figure 1.3: A trajectory of the billiard flow in a rectangle: elastic reflections according to the law of optics.

If the trajectory hits a *corner* of the boundary, it ends there (as when a ball hit a pocket in a billiard table). In a real billiard game, one is interested in sending balls to a pocket. In a mathematical billiard, though, one considers only in trajectories which *do not* hit any pocket, and, since the billiard is frictionless, continue their motion forever. One is then interested in describing dynamical properties, for example asking whether trajectories are dense, or periodic, or trapped in certain regions of the billiard (see for example the circle billiard description in § 1.2.2 below).

Let $X = D \times [0, 2\pi)$ be the set of pairs (x, θ) where $x \in D$ is a point in D and $\theta \in [0, 2\pi)$ a direction[1]. The billiard flow $b_t : X \to X$ sends (x, θ) to the point $b_t(x, \theta) = (x', \theta')$, where x' is the point reached after time t moving with constant unit speed along the billiard trajectory from x in direction θ and θ' is the new direction after time t.

The last example if the more general set up in which flows appear.

Example 1.1.3. [Solutions of differential equations] Let $X \subset \mathbb{R}^n$ be a space, $g : X \to \mathbb{R}^n$ a function, $x_0 \in X$ an initial condition and

$$\begin{cases} \dot{x}(t) = g(x) \\ x(0) = x_0 \end{cases} \tag{1.1}$$

be a differential equation. If the solution $x(x_0, t)$ is well defined for all t and all initial conditions $x_0 \in X$, if we set $f_t(x_0) := x(x_0, t)$ we have an example of a continuous dynamical system. In this case, an orbit is given by the trajectory described by the solution:

$$\mathcal{O}_{f_t}(x) := \{x(x_0, t), \quad t \in \mathbb{R}\}.$$

Exercise 1.1.1. Verify that $f_t(x_0) := x(x_0, t)$ defined as above satisfies the properties of a flow.

Dynamical systems as actions

A more formal way to define a dynamical system is the following, using the notion of *action*.

Let X be a space and G group (as $\mathbb{Z}$ or $\mathbb{R}$ or $\mathbb{R}^d$) or a semigroup (as $\mathbb{N}$).

[1]To be more precise, one should consider as X the space of paris (x, θ) where if x belongs to the interior of D the angle can be any $\theta \in [0, 2\pi)$, but if x belongs to the boundary of D, the direction θ is contrained to pointing inwards.

Definition 1.1.2. *An* action *of G on X is a map $\psi : G \times X \to X$ such that, if we write $\psi(g, x) = \psi_g(x)$ we have*

(1) If e id the identity element of G, $\psi_e : X \to X$ is the identity map;

(2) For all $g_1, g_2 \in G$ we have $\psi_{g_1} \circ \psi_{g_2} = \psi_{g_1 g_2}$.[2]

A discrete dynamical systems is then defined as an action of the group $\mathbb{Z}$ or of the semigroup $\mathbb{N}$. A continuous dynamical system is an action of $\mathbb{R}$. There are more complicated dynamical systems defined for example by actions of other groups (for example $\mathbb{R}^d$).

Exercise 1.1.2. Prove that the iterates of a map $f : X \to X$ give an action of $\mathbb{N}$ on X. The action $\mathbb{N} \times X \to X$ is given by

$$(n, x) \to f^n(x).$$

Prove that if f is invertible, one has an action of $\mathbb{Z}$.

Exercise 1.1.3. Prove that the solutions of a differential equation as (1.1) (assuming that for all points $x_0 \in X$ the solutions are defined for all times) give an action of $\mathbb{R}$ on X.

Ergodic properties of flows

The ergodic theory of continuous time dynamical systems can be developed similarly to the discrete time theory. The main ergodic theory definitions for flows are very similar to the corresponding ones for maps:

Assume that $(X, \mathscr{B}, \mu)$ is a measured space.

Definition 1.1.3. *A flow $f_t : X \to X$ is* measurable *if for any $t \in \mathbb{R}$, f_t is a measurable transformation and furthermore the map $F : X \times \mathbb{R} \to X$ given by $F(x, t) = f_t(x)$ is measurable.*

A measurable flow $f_t : X \to X$ is measure-preserving *and preserves the measure μ if for any $t \in \mathbb{R}$, f_t preserves μ, that is for any measurable set $A \in \mathscr{B}$, $\mu(f_t^{-1}(A)) = \mu(A)$ transformation.*

Remark that since each transformation f_t in a flow is invertible, one can equivalently ask that $\mu(f_t(A)) = \mu(A)$ for all $A \in \mathscr{B}$ and $t \in \mathbb{R}$.

Definition 1.1.4. *A measurable set $A \in \mathscr{B}$ is* invariant *under $f_t : X \to X$ if*

$$f_t(A) = f_t^{-1}(A) = A$$

for all $t \in \mathbb{R}$.

A measure-preserving flow $f_t : X \to X$ on a probability space is ergodic *if for any $t \in \mathbb{R}$, f_t is ergodic, that is if $A \in \mathscr{B}$ is invariant under the flow, then either $\mu(A) = 0$ or $\mu(A) = 1$.*

The statement of the Birkoff ergodic theorem for continuous time dynamical systems is also similar but with the difference that the time average involves an *integral* instead than a *sum*: if $g : X \to \mathbb{R}$ is an observable (that is an integrable function) the time average of the observable g along the trajectory $\mathcal{O}^+_{f_t}(x)$ of x under $f_t : X \to X$ up to time T is given by

$$\frac{1}{T} \int_0^T g(f_t(x)) \mathrm{d}x.$$

[2] If X has an additional structure (for example X is a topological space or X is a measured space), we can ask the additional requirement that for each $g \in G$, $\psi_g : X \to X$ preserves the structure of X (for example ψ_g is a continuous map if X is a topological space or ψ_g preserves the measure. We will see more precisely these definitions in Chapters 2 and 4.

[Recall that if $f : X \to X$ is a transformation, the time average of the observable g along the trajectory $\mathcal{O}_f^+(x)$ of x under $f : X \to X$ up to time N is given by

$$\frac{1}{N}\sum_{n=0}^{N-1} g(f^n(x)).]$$

Theorem 1.1.1 (Birkohff Ergodic Theorem for Continuous time)**.** *If $(X, \mathscr{B}, \mu)$ is a probability space and $f_t : X \to X$ is an* ergodic flow*, for any $g \in L^1(X, \mu)$ and μ-almost every $x \in X$ the following limit exists and we have*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T g(f_t(x))\mathrm{d}x = \int_X g\mathrm{d}\mu,$$

that is the time averages of g converge to the space average of g for μ-almost every initial point $x \in X$.

The definition of mixing is again very similar:

Definition 1.1.5. *A measure-preserving flow $f_t : X \to X$ on a probability space is* mixing *if any pairs of measurable sets $A, B \in \mathscr{B}$ one has*

$$\lim_{t\to\infty} \mu(f_t^{-1}(A) \cap B) = \mu(A)\mu(B),$$

or, equivalently, since f_t is invertible,

$$\lim_{t\to\infty} \mu(f_t^{(}A) \cap B) = \mu(A)\mu(B).$$

1.2 Two examples of Poincaré maps

In many examples of continuous time dynamical systems, a procedure first used by Poincaré allows to reduce the continuous dynamical system to a discrete dynamical systems, by considering what is nowadays called *Poincaré map.* (Poincaré, who can be considered the father of dynamical systems, used the *Poincaré map* to study the motion of the solar system). Here below we give two special examples of flows (linear flows on $\mathbb{T}^2$ and the billiard flow in a circle) for which the Poincaré map turns out to be a rotation of a circle.

1.2.1 Poincaré maps of linear flows on the torus.

Let $X = \mathbb{T}^2$ and let $\varphi_t^\theta : \mathbb{T}^2 \to \mathbb{T}^2$ be the linear flow in direction θ. Assume that $\theta \neq \pi/2$, that is assume that the flow is *not* vertical.

Let $\Sigma \subset X$ be the curve in $\mathbb{T}^2$ obtained considering the vertical side $0 \times [0, 1) \subset [0, 1)^2$ of the unit square representing $\mathbb{T}^2$. Notice that the vertical side gives a *closed* curve on $\mathbb{T}^2$, since the endpoints are identified. On the surface of a doughnut, Σ is one of the meridians of the torus (see the dark curve in Figure 1.1(b)). The curve Σ is *transverse* to the flow, that is for any $y \in \Sigma$ the trajectory $\mathcal{O}^+_{\varphi_t^\theta}(y)$ is *not* contained in the curve, but moves away from it (since we are assuming that the curve is vertical but the flow is *not* vertical). The curve Σ is called a *transverse section.* We will identify Σ with a unit interval $[0, 1)$ and use the coordinate y to denote points of Σ to remember that y is the vertical coordinate in the square.

Consider the map $T : \Sigma \to \Sigma$ obtained as follows: given $y \in \Sigma$, follow the trajectory $\mathcal{O}^+_{\varphi_t^\theta}(y)$ until it hits the curve Σ again (see the darker part of a trajectory in Figure 1.1(b)). Let $T(y)$ be the *first return* to Σ, that is let t_y be the minimum $t > 0$ such that $\varphi_t^\theta(y) \in \Sigma$ and set

$$T(y) = \varphi_{t_y}^\theta(y).$$

The map T is called *Poincaré map* or *first return map* of the flow to Σ.

Let us prove that the map $T : \Sigma \to \Sigma$ is a *rotation*, that is

$$T(y) = R_\alpha(y) = y + \alpha \mod 1, \qquad \text{where} \qquad \alpha = \tan\theta.$$

The trajectory of $\varphi_t^\theta(x)$ starting from from $y_n \in \Sigma$, for small $t > 0$, is simply a line in direction θ. Remark that the other vertical side $1 \times [0, 1)$ of the square also projects to the same curve in $\mathbb{T}^2$.

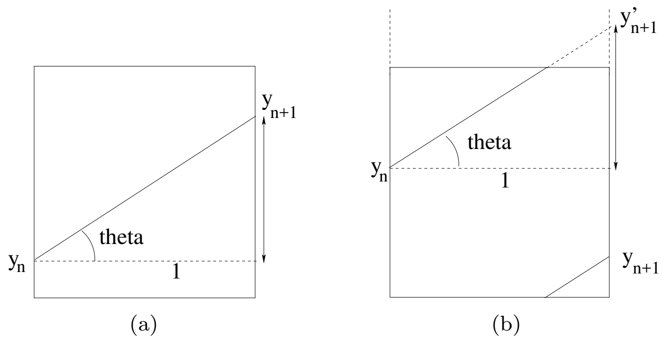


Figure 1.4: The first return y_{n+1} of y_n to the vertical sides representing Σ.

Thus, if the trajectory from $y_n \in \Sigma$ does not hit the top boundary of the square $[0, 1)^2$ before returning to Σ, the first return $y_{n+1} = T(y_n)$ is simply the first point when this line hits the right vertical side of the square. This point can be computed by simple trigonometry (see Figure 1.4(a)): $y_{n+1} - y_n$ is the length of a triangle with an angle θ and one side of length 1, thus

$$\tan\theta = \frac{y_{n+1} - y_n}{1} = T(y_n) - y_n \qquad \Rightarrow \qquad T(y_n) = y_n + \tan\theta.$$

If the trajectory from $y_n \in \Sigma$ hits the boundary of the square $[0, 1)^2$ before returning to Σ, let us draw two copies of the square on top of each other: let y'_{n+1} be the first point where the line of slope θ through x hit the vertical line through $(1, 0)$. This point can be computed by simple trigonometry as above (see Figure 1.4(b)) and is given by

$$y'_{n+1} = y_n + \tan\ theta.$$

After hitting the top boundary of the square $[0, 1)^2$, the trajectory of y_n continues coming out from the lower side of the square $[0, 1)^2$ and consists of copies of the segment in direction θ translated by integer vectors. For example in Figure 1.4(b) the segment of trajectory after hitting the top boundary is obtained translating by $(0, -1)$ the segment in the second copy of the square. Thus, the first return $T(y_n)$ is obtained by translating y'_{n+1} by an integer:

$$T(y_n) = y'_{n+1} \mod 1 = y_n + \tan\theta \mod 1.$$

This shows exactly T is the rotation R_α by $\alpha = \tan\theta$.

As an application, let us show:

Theorem 1.2.1. *If θ is* irrational*, all the orbits $\mathcal{O}^+_{\varphi_t^\theta}(x)$, $x \in \mathbb{T}^2$, of the linear flow φ_t^θ in direction θ are* dense *in $\mathbb{T}^2$. In other words, irrational linear flows are* minimal.

Proof. Let $x \in \mathbb{T}^2$. To show that $\mathcal{O}^+_{\varphi_t^\theta}(x)$ is dense, we have to show that for each open set $U \subset \mathbb{T}^2$ the trajectory of x visits U, that is there exists $t \geq 0$ such that $\varphi_t^\theta(x) \in U$.

Let us consider the *projection* of U in direction of θ to the vertical side Σ (see Figure 1.5) and let us call $I \subset \Sigma$ the interval obtained as image of such projection. By construction, any line in direction θ starting from a point $y \in I$ enters U. Thus, it is enough to show that the trajectory $\mathcal{O}^+_{\varphi^\theta_t}(x)$ visits I, since if $\varphi^\theta_t(x) \in I$ for some $t > 0$, for a successive $t' > t$ we also have $\varphi^\theta_{t'}(x) \in U$.

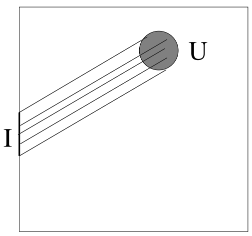


Figure 1.5: Projection $I \subset \Sigma$ of an open set U in direction of the flow.

Since $I \subset \Sigma$, the visits of the trajectory $\mathcal{O}^+_{\varphi^\theta_t}(x)$ to I are a subset of the visits to Σ. If y_0 is the first point of $\mathcal{O}^+_{\varphi^\theta_t}(x)$ that visits Σ, the next visits are $T(y_0)$, $T^2(y_0), \dots, T^n(y_0), \dots$.

We prove in the second lecture that irrational rotations are minimal. Since θ is irrational, $\alpha = \tan\theta$ is irrational and $T = R_\alpha$ is minimal. Thus, in particular, the orbit of y_0 is dense, which implies that there exists n_0 such that $T^{n_0}(y_0) \in I$. This means that $\mathcal{O}^+_{\varphi^\theta_t}(x)$ visits I and hence it visits U and concludes the proof that $\mathcal{O}^+_{\varphi^\theta_t}(x)$ is dense. $\square$

Exercise 1.2.1. Show that if $\tan\theta$ is *rational* all the orbits of $\varphi^\theta_t : \mathbb{T}^2 \to \mathbb{T}^2$ are closed curves.

1.2.2 Poincaré maps of a billiard in the circle.

Let $D \subset \mathbb{R}^2$ be the interior plus the circumference S^1 of a circle of radius 1. One can show using simple Euclidean geometry (see Exercise 1.2.2 and Figure 1.6(b)) that any billiard trajectory in a circle is always tangent to a inner circle of some radius $0 < d < 1$. More precisely, given a point $x \in D$ and a direction θ, consider the billiard trajectory $\{b_t(x,\theta), t \geq 0\}$ consists of consecutive segments all tangent to a circle of radius $d = \cos\theta$ (see Figure 1.6(a)). Thus, it is clear that trajectories *cannot* be dense, since they are *trapped* in an annular region A_θ bounded by the circle of radius d and the circle of radius r, as shown in Figure 1.6(a).

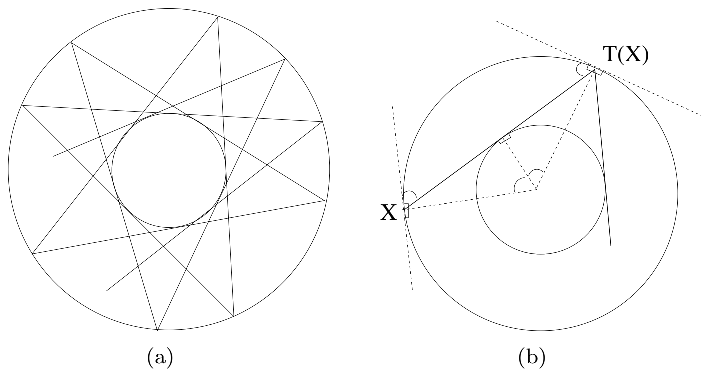


Figure 1.6: Billiard trajectories in a circle billiard.

Nevetheless, one can still ask whether the trajectories of the circular billiard are *dense* in the annulus that traps them. We will see that if θ is *rational*, trajectories are closed, while if θ is *irrational*, billiard trajectories in direciton θ are dense in A_θ.

Let $\Sigma = S^1$ be the circle boundary of the billiard. Fix a direction θ. The *Poincaré* or *first return map* $T : \Sigma \to \Sigma$ of the billiard to Σ is the map that sends $x \in \Sigma$ to the point $T(x) \in \Sigma$ in which the billiard trajectory $\{b_t(x, \theta), t \geq 0\}$ of x forming an angle θ with the tangents first hits Σ again, as in Figure 1.6(b). Again by Euclidean geometry (see Exercise 1.2.2) one can see that $T : \Sigma \to \Sigma$ is again *rotation of the circle* by an angle $\alpha = 2\theta$. Thus if we identify $\Sigma = S^1$ with the unit interval $[0,1]/\sim$, for any $x \in [0,1)$

$$T(x) = x + 2\theta \mod 1.$$

As we saw in Lecture 2, if θ is *rational* (and hence $\alpha = 2\theta$ is also rational), all orbits of T are *periodic* (of period q if $\alpha = p/q$ with p, q coprime). If the orbit of x under T is periodic of period n, the billiard trajectory closes up after hitting the boundary n times. Thus, if θ is *rational*, all orbits of the circular billiards are closed.

On the other hand, if θ (and hence α) is *irrational*, the rotation $R_2\theta$ is minimal, thus all orbits of $T = R_{2\theta}$ are *dense* in S^1. Let us deduce that all billiard trajectories as in Figure 1.6(a) are *dense* in the annulus A_θ. Let $U \in A_\theta$ be an open set in the annulus, as in Figure 1.6(c). Draw all lines that start from a points of D and are tangent to the inner circle of radius $d = \cos\theta$, as in Figure 1.6(c). This lines hit the circle S^1 in a interval I. By construction, if $x \in I$, the billiard trajectory starting from x and forming and angle θ with S^1 will visit the open set U.

Thus, it is enough to consider successive points in which the billiard trajectory forming an angle θ with the tangents hits the boundary $\Sigma = S^1$ and show that one of this points belong to I, since this implies that that the trajectory will then visit U. If $x \in S^1$ is the first time the trajectory hits the boundary, the successive times are given by $T(x), T^2(x), \ldots$. Since T is minimal, there exists $n \in \mathbb{N}$ such that $T^n(x) \in I$. This concludes the proof.

Exercise 1.2.2. Let S^1 be a circle of radius one and consider a billiard trajectory inside S^1.

(i) Let $x \in S^1$ be a point of the trajectory and assume that the angle of incidence between the trajectory and the tangent to S^1 at x is θ. Show that if $T(x)$ is the following point in which the trajectory hits S^1 the angle of incidence is again θ. [Figure 1.6(a) might be useful: right angles and angles equal to θ are marked.]

(ii) Deduce from (i) that any billiard trajectory which hits S^1 forming an angle θ with the tangents remains tangent to a inner circle of radius $d = \cos\theta$;

(ii) Deduce from (i) that the first return map $T : S^1 \to S^1$ is a rotation by $\alpha = 2\theta$.

Extra: Poincaré maps more in general.

Poincaré maps can be defined more in general for many continuous time dynamical systems. Given a flow $f_t : X \to X$, a *section* $\Sigma \subset X$ is a subset of one dimension less than the ambient space (for example, when X is a surface, as in the case $X = \mathbb{T}^2$, a section Σ is a curve; if X is 3-dimensional, a section Σ is a surface, as in Figure 1.7, and so on).

Assume for example that f_t is *minimal*, that is all trajectories $\mathcal{O}^+_{f_t}(x)$, $x \in X$ are *dense*. Assume also that there exists a section Σ *transverse* to the flow, that is, for any point $x \in \Sigma$, $f_t(x)$ does not belong to Σ for all sufficiently small $t > 0$. In other words, Σ is transverse to f_t if all trajectories starting from Σ flow *through* it and are not contained in it. In this case we can define a map $T : \Sigma \to \Sigma$ which is the Poincaré map or *first return map* of f_t to Σ.

Indeed, one can show using minimality that all points $x \in \Sigma$ *return* to Σ, that is for each $x \in \Sigma$ there is a first return time $t_x > 0$ such that

$$f_{t_x}(x) \in \Sigma,$$

and is the *first* return time, that is $f_t(x) \notin \Sigma$ for all $0 < t < t_x$. In this case, the Poincaré map $T : \Sigma \to \Sigma$ is well defined and given by

$$T(x) = f_{t_x}(x).$$

There are weaker assumptions that allow to define the Poincaré map on a transversal section $\Sigma \subset X$ of the flow. For example, if the flow preserves a *finite* measure, the Poincaré recurrence theorem holds so that the trajectory of $\mu-$almost every $x \in X$ is *recurrent*. One can use recurrence to show that the first return map $T : \Sigma \to \Sigma$ is well defined for *almost every*[3] $x \in \Sigma$

If x_0 is a periodic point for the flow $f_t : X \to X$, that is there exists t_0 such that $f_{t_0}(x) = x$, then one can choose a small section Σ containing the point x_0 and define the Poincaré map $x \in \Sigma$ for points near x_0. This was the original set up studied by Poincaré.

Figure 1.7: Poincaré map P on a surface section Σ in dimension 3.

1.3 Poincaré Hopf method to prove ergodicity

Historically, shorting after the notion of ergodicity was introduced and the ergodic theorems (by von Neumann and Birkhoff) were proved, one of the first examples of proofs of ergodicity was given by Hopt, for a fundamental geometric examples, the hyperbolic geodesic flow. In the next section we will give the algebraic description of this flow, then explain the arguments in Hopf proof of ergodicity. His method of proof is know called *Hopf argument* for ergodicity and has proved to be very useful in proving ergodicity in a great variety of dynamical systems (mostly those which exhibit some for of *hyperbolicity*.

1.3.1 The geodesic flow (algebraic description)

We saw flows given by solutions of differential equations and flows described geometrically (like the billiard flow). Another way of describing flows is through group actions on group of matrices. We will see in this section an important example of *algebraic* flow, namely the (hyperbolic) *geodesic flow* (and, connected to it, also the *horocycle flow*. This is a fundamental example in *homogeneous dynamics*.

Let $G = SL(2, \mathbb{R})$ be the set of 2×2 matrices with real entries and determinant one, i.e.

$$G = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}, \quad det(g) = ad - bc = 1. \right\}$$

[3]Here *almost every* refers to a measure on Σ obtained restricting μ to Σ.

Then G is a group, with matrix multiplication as a group operation (it is actually a *topological group*, i.e. a group with a *topology*, so that the group operation is continuous). One can consider a distance on G, for example given by the sum of the distances (in $\mathbb{R}$) between entries.

Any 1−parameter subgroup of G defines a flow by matrix multiplication. We will be interested in the subgroup A of diagonal matrices, that we can write as

$$A = \left\{ a_t := \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \quad t \in \mathbb{R}. \right\}$$

Then, the *geodesic flow* (also called *diagonal flow*) $g_t : G \to G$ is given by

$$g_t(g) = g \cdot a_t = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} = \begin{pmatrix} ae^{t/2} & be^{-t/2} \\ ce^{t/2} & de^{-t/2} \end{pmatrix}$$

The flow $g_t : G \to G$ has a natural invariant measure.

Definition 1.3.1. *The* Haar measure μ *on* G *is the measure given by integrating the density* $f(g)$ *given by*

$$f(g) = \frac{xyz}{x}, \qquad \text{if} \quad g = \begin{pmatrix} x & y \\ z & w \end{pmatrix},$$

so that, if we identify G *with* $\mathbb{R}^3$ *(remark that given the entries* x, y, z *of* g *the last entry* w *is determined by the determinant, namely* $w = yz/x$.

[More in general, for any topological group G, Haar proved the existence of a measure which is invariant under right multiplication; these measure is now called *Haar measure* and this definition simply gives the explit expression of this measure for $G = SL(2, \mathbb{R})$.]

We have the following result (which we will not prove):

Lemma 1.3.1. *The Haar measure* μ *is* invariant *under the geodesic flow* $g_t : G \to G$.

The measure μ is however *infinite* (so not good for the point of view of ergodic theory, i.e. to apply Poincaré ergodic theorem or to investigate ergodicity and mixing). For this reason, we want to consider a *quotient space* of G. What we will now do is similar to what we do when we start from the Lebesgue measure of $\mathbb{R}^2$, which is infinite: to get a *finite* invariant measure, one can consider the torus $X = \mathbb{R}^2/\mathbb{Z}^2$ which is the quotient of $\mathbb{R}^2$ (which is a topological group) by the (*discrete*) subgroup $\mathbb{Z}^2$.

Let us hence consider a *discrete* subgroup $\Gamma < G$ (discrete here means that each element $g \in \Gamma$ has a neighbourhood that does not contain any other element of Γ. We will take $\Gamma = SL(2, \mathbb{Z})$, where $SL(2, \mathbb{Z}) < SL(2, \mathbb{R})$ are matrices of the same form but with *integer* entries:

$$\Gamma = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z}, \quad det(g) = ad - bc = 1. \right\}$$

Let us consider as space X the quotient space $\Gamma \backslash G$, i.e. the space of *left cosets*: $X := \Gamma \backslash G = \{x := \Gamma g, \quad g \in G\}$. In other words, points $x \in X$ are equivalence classes of elements of G so that $g \equiv g'$ iff $g = \gamma g'$ for some $\gamma \in \Gamma$. We then write $x = \Gamma g$ for the equivalence class of an element g.

The Haar measure μ on G gives also a measure on the quotient X, which we still denote by μ (analogously to the Lebesgue measure λ on $\mathbb{R}^2$ which also gives a Lebesgue measure λ on the torus $X = \mathbb{R}^2/\mathbb{Z}^2$).

Lemma 1.3.2. *The Haar measure* μ *on* $X = SL(2, \mathbb{Z}) \backslash SL(2, \mathbb{R})$ *if* finite. *More precisely one can show that* $\mu(X) = 2\pi/2$.

We will hence renormalize μ to be a probability measure, i.e. define $\overline{\mu} = \frac{2}{3\pi}\mu$ to be the *normalized Haar measure*, so that $\overline{\mu}$ is a probability measure ($\overline{\mu}(X) = 1$).

We then have the following fundamental result:

Theorem 1.3.1 (Hopf). *The geodesic flow $g_t : X \to X$ is* ergodic *with respect to the normalized Haar measure $\overline{\mu}$.*

The theorem was proved by Hopf. The proof of this theorem will be given in the next section. As we said at the beginning, the method of the proof is known as *Hopf argument* and was used to prove ergodicity in many (hyperbolic) dynamical systems.

1.3.2 Poincaré Hopf method to prove ergodicity

This section will be added soon. In the meanwhile, a good reference for Hopf arguments are the following notes online (see section 7):

https://homepages.warwick.ac.uk/ masdbl/ergodictheory-1May2011.pdf