MAT733 - HS2018

Dynamical Systems and Ergodic Theory

Part III: Ergodic Theory

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Chapter 3

Ergodic Theory

In this last part of our course we will introduce the main ideas and concepts in *ergodic theory.* Ergodic theory is a branch of dynamical systems which has strict connections with analysis and probability theory. The discrete dynamical systems $f : X \to X$ studied in topological dynamics were continuous maps f on metric spaces X (or more in general, topological spaces). In ergodic theory, $f : X \to X$ will be a *measure-preserving* map on a *measure space* X (we will see the corresponding definitions below). While the focus in topological dynamics was to understand the qualitative behavior (for example, periodicity or density) of *all* orbits, in ergodic theory we will not study *all* orbits, but only *typical*[1] *orbits*, but will investigate more *quantitative* dynamical properties, as frequencies of visits, equidistribution and mixing.

An example of a basic question studied in ergodic theory is the following. Let $A \subset X$ be a subset of the space X. Consider the visits of an orbit $\mathscr{O}_f^+(x)$ to the set A. If we consider a finite orbit segment $\{x, f(x), \dots, f^{n-1}(x)\}$, the number of visits to A up to time n is given by

$$Card \{ \ 0 \le k \le n-1, \qquad f^k(x) \in A \ \}. \tag{3.1}$$

A convenient way to write this quantity is the following. Let χ_A be the characteristic function of the set A, that is a function $\chi_A : X \to \mathbb{R}$ given by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Consider the following sum along the orbit

$$\sum_{k=0}^{n-1} \chi_A(f^k(x)). \tag{3.2}$$

This sum gives exactly the number (3.1) of visits to A up to time n. This is because $\chi_A(f^k(x)) = 1$ if and only if $f^k(x) \in A$ and it is zero otherwise, so that there are as many ones in the sum in (3.2) than visits up to time n and summing them all up one gets the total number of visits up to time n.

If we divide the number of visits up to time n by the time n, we get the *frequency* of visits up to time n, that is

$$\frac{Card\{0 \le k < n, \quad \text{such that } f^k(x) \in X\}}{n} = \frac{1}{n} \sum_{k=0}^{n-1} \chi_A(f^k(x)).$$

The frequency is a number between 0 and 1.

Q1 Does the frequency of visits converge to a limit as n tends to infinity? (for all points? for a typical point?)

[1]Typical will become precise when we introduce measures: by typical orbit we mean the orbit of *almost every* point, that is all orbits of points in a set of full measure.

Q2 If the limit exists, what does the frequency tend to?

A useful notion to consider for dynamical systems on the circle (or on the unit interval) is that of uniform distribution, that we have already encountered in Chapter 1.

Recall that given a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,1]$ (or $\mathbb{R}/\mathbb{Z}$), we say that $\{x_n\}_{n\in\mathbb{N}}$ is uniformly distributed if for all intervals $I \subset [0,1]$ we have that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_I(x_k) = |I|.$$

Where $|I|$ denotes the length of the interval $|I|$. An equivalent definition (as we saw in the formulation of the Weyl's criterion) is that for all continuous functions $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ we should have

$$\frac{1}{n} \sum_{k=0}^{n-1} f(x_k) = \int_0^1 f(x)dx.$$

So if we have a dynamical system $T : [0,1] \to [0,1]$ (or $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$) we can ask whether orbits $\{x, T(x), T^2(x), \ldots\}$ are uniformly distributed or not.

Example 3.0.1. Let $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be the rotation given by $R_\alpha(x) = x + \alpha$. If $\alpha \in \mathbb{Q}$ then for all $x \in \mathbb{R}/\mathbb{Z}$ the orbit of R_α will be periodic, so cannot be dense and thus cannot be uniformly distributed (why?). On the otherhand if $\alpha \notin \mathbb{Q}$, for all $x \in \mathbb{R}/\mathbb{Z}$ the orbit of R_α is uniformly distributed, as we proved in Chapter 1 using Weyl's criterion.

The result above about rotations is often thought of as the first ergodic theorem to have been proved and was proved independently in 1909 and 1910 by Bohl, Sierpiński and Weyl.

A more complicated example is the following:

Example 3.0.2. Let $T : [0,1) \to [0,1)$ be the doubling map given by $T(x) = 2x \mod 1$. We know that there is a dense set of x for which the orbit $\mathcal{O}_T^+ x$ is periodic and hence not uniformly distributed. However it will turn out the for "almost all" x the orbit of T is uniformly distributed (where almost all can be thought of as meaning except for a set of 0 length, or in other words of zero Lebesgue measure).

We may also have maps $T : [0,1) \to [0,1)$ where for 'typical' x orbits are not equidistributed

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} X_A(T^i(x)) = \int_A f(x)dx$$

for some suitable function f (we will see that the Gauss map is an example of such a map). To make these notions precise we need to introduce some measure theory which will have the additional advantage of introducing a theory of integration which is more suited to our purposes.

Spaces and transformations in different branches of dynamics

Measure spaces and measure-preserving transformations are the central object of study in ergodic theory. Different branches of dynamical systems study dynamical systems with different properties. In *topological dynamics*, the discrete dynamical systems $f : X \to X$ studied are the ones in which X is a metric space (or more in general, a topological space) and the transformation f is continuous. In *ergodic theory*, the discrete dynamical systems $f : X \to X$ studied are the ones in which X is a *measured space* and the transformation f is *measure-preserving*.

Similarly, other branches of dynamical systems study spaces with different structures and maps which preserves that structure (for example, in *holomorphic dynamics* the space X is a subset of the complex plan $\mathbb{C}$ (or $\mathbb{C}^n$) and the map $f : X \to X$ is a *holomorphic map*; in *differentiable dynamics* the space X is a subset of $\mathbb{R}^n$ (or more in general a *manifold*, for example a surface) and the map $f : X \to X$ is *smooth* (that is differentiable and with continuous derivatives) (as summarized in the Table below) and so on...

branch of dynamics	**space** X	**transformation** $f : X \to X$
Topological dynamics	metric space (or topological space)	continuous map
Ergodic Theory	measure space	measure-preserving map
Holomorphic Dynamics	subset of $\mathbb{C}$ (or $\mathbb{C}^n$)	holomorphic map
Smooth Dynamics	subset of $\mathbb{R}^n$ (or manifold, as surface)	smooth (continuous derivatives)

3.1 Review of measures and the Extension Theorem

In this section we review basic notions on measures and measure spaces, with a particular emphasis on the examples of measures and measurable spaces that we will use in this Chapter. We in particular state the Extension theorem and show how it will be implicitely used in the following sections to define measures and prove that certain transformations are measure preserving.

A *measure* μ on a space X is a function from a collection of subsets of X, called *measurable sets*, which assigns to each measurable set A its measure $\mu(A)$, that is a positive number (possibly infinity). You already know at least two natural examples of measures.

Example 3.1.1. Let $X = \mathbb{R}$. The 1-dimensional Lebesgue measure λ on $\mathbb{R}$ assigns to each interval $[a, b] \in \mathbb{R}$ its *length*:

$$\lambda([a, b]) = b - a, \qquad a, b \in \mathbb{R}.$$

Let $X = \mathbb{R}^2$. The 2-dimensional Lebesgue measure, that we will still call λ, assigns to each measurable set[2] $A \subset \mathbb{R}^2$ its *area*, which is given by the integral[3]

$$\lambda(A) = \text{Area}(A) = \int_A \mathrm{d}x\mathrm{d}y.$$

Measurable spaces

One might hope to assign a measure to *all* subsets of X. Unfortunately, if we want the measure to have the reasonable and useful properties of a measure (listed in the definition of measure below), this leads to a contradiction (see Extra if you are curious). So, we are forced to assign a measure only to a sub-collection all subsets of X. We ask that the collection of measurable subsets is closed under the operation of taking countable unions in the following sense.

Definition 3.1.1. A collection $\mathscr{A}$ of subsets of a space X is called an *algebra* of subsets if

(i) The empty set $\emptyset \in \mathscr{A}$;

(ii) $\mathscr{A}$ is closed under complements, that is if $A \in \mathscr{A}$, then its complement $A^c = X \backslash A$ also belongs to $\mathscr{A}$;

(iii) $\mathscr{A}$ is closed under *finite* unions, that is if $A_1, \ldots, A_n \in \mathscr{A}$, then

$$\bigcup_{i=1}^{n} A_i \in \mathscr{A}.$$

[2]We will precisely define what are the measurable sets for the Lebesgue measure in what follows.

[3]If A is such that χ_A is integrable in the sense of Riemann, this integral is the usual Riemannian integral. More in general, we will need the notion of Lebesgue integral, which we will introduce in the following lectures.

Example 3.1.2. If $X = \mathbb{R}$ an example of algebra is given by the collection $\mathscr{A}$ of all possible finite unions of subintervals of $\mathbb{R}$.

Exercise 3.1.1. Check that the collection $\mathscr{A}$ of all possible finite unions of subintervals of $\mathbb{R}$ is an algebra.

Definition 3.1.2. A collection $\mathscr{A}$ of subsets of a space X is called a *$\sigma-$algebra* of subsets if

(i) The empty set $\emptyset \in \mathscr{A}$;

(ii) $\mathscr{A}$ is closed under complements, that is if $A \in \mathscr{A}$, then its complement $A^c = X\backslash A$ also belongs to $\mathscr{A}$;

(iii) $\mathscr{A}$ is closed under *countable* unions, that is if $\{A_n, \quad n \in \mathbb{N}\} \subset \mathscr{A}$, then

$$\bigcup_{n=1}^{\infty} A_n \in \mathscr{A}.$$

Thus, a $\sigma-$algebra is an algebra which in addition is closed under the operation of taking countable unions. The easiest way to define a $\sigma-$algebra is to start from any collection of sets, and take the closure under the operation of taking complements and countable unions:

Definition 3.1.3. If $\mathcal{S}$ is a collection of subsets, we denote by $\mathscr{A}(\mathcal{S})$ the smallest $\sigma-$algebra which contains $\mathcal{S}$. The *smallest* means that if $\mathscr{B}$ is another $\sigma-$algebra which contains $\mathcal{S}$, then $\mathscr{A}(\mathcal{S}) \subset \mathscr{B}$. We say that $\mathscr{A}(\mathcal{S})$ is the $\sigma-$algebra *generated by* $\mathcal{S}$.

The following example/definition is the main example that we will consider.

Definition 3.1.4. If (X, d) is a metric space[4], the *Borel $\sigma-$algebra* $\mathscr{B}(X)$ (or simply $\mathscr{B}$) is the smallest $\sigma-$algebra which contains all *open sets* of X. The subsets $B \in \mathscr{B}(X)$ are called *Borel sets.*

Borel σ-algebras are the natural collections of subsets to take as measurable sets. In virtually all of our examples, the measurable sets will be Borel subsets.

Definition 3.1.5. A *measurable space* $(X, \mathscr{A})$ is a space X together with a $\sigma-$algebra $\mathscr{A}$ of sets. The sets in $\mathscr{A}$ are called *measurable sets* and $\mathscr{A}$ is called the $\sigma-$algebra of measurable sets.

Example 3.1.3. If (X, d) is a metric space, $(X, \mathscr{B}(X))$ is a measurable space, where $\mathscr{B}(X)$ is the Borel $\sigma-$algebra.

Measures

We can now give the formal definition of measure.

Definition 3.1.6. Let $(X, \mathscr{A})$ be a measurable space. A measure μ is a function $\mu : \mathscr{A} \to \mathbb{R}^+ \cup \{+\infty\}$ such that

(i) $\mu(\emptyset) = 0$;

(ii) If $\{A_n, \quad n \in \mathbb{N}\} \subset \mathscr{A}$ is a countable collection of *pairwise disjoint* measurable subsets, that is if $A_n \cap A_m = \emptyset$ for all $n \neq m$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n). \tag{3.3}$$

We say that the measure μ is *finite* if $\mu(X) < \infty$.

Remark that to have (3.3) we need to assume that A_n are *disjoint.*

[4]The same definition of Borel $\sigma-$algebra holds more in general if X is a topological space, so that we know what are the open sets.

Example 3.1.4. You can check that both length and area have this natural property: for example the area of the union of disjoint sets is the sum of the areas. If $X = \mathbb{R}$, the Lebesgue measure on $\mathbb{R}$ is *not* finite, since $\lambda(\mathbb{R}) = +\infty$. On the contrary, the Lebesgue measure restricted to an interval $X = [a, b] \subset \mathbb{R}$ is finite since $\lambda([a, b]) < \infty$. Similarly, if $\mathbb{T}^2$ is the torus and we consider the area λ, $Area(\mathbb{T}^2) = \lambda(\mathbb{T}^2) = 1 < \infty$, so λ is a finite measure on $\mathbb{T}^2$.

Definition 3.1.7. A *measure space* $(X, \mathscr{A}, \mu)$ is a measurable space $(X, \mathscr{A})$ and a measure $\mu : \mathscr{A} \to \mathbb{R}^+ \cup \{+\infty\}$.

If $\mu(X) = 1$, we say that $(X, \mathscr{A}, \mu)$ is a *probability space.*

If we just work directly from the definition of a measure it is hard to produce examples of measures. One simple example is the following (as well as being simple it also turns out to be extremely useful).

Example 3.1.5. Let X a space and $x \in X$ a point. In this example we can take $\mathscr{A}$ to be the collection of all subsets of X. The measure δ_x, called *Dirac measure at* x, is defined by

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Thus, the measure δ_x takes only two values, 0 and 1, and assigns measure 1 only to the sets which contain the point x.

It is also straight forward to see that if $(X, \mathscr{A})$ is a measurable space and μ_1, μ_2 are measures on $(X, \mathscr{A})$ them $\mu_1, \mu_2 : \mathscr{A} \to \mathbb{R}^+ \cup \{+\infty\}$ given by

$$\mu_1 + \mu_2(A) = \mu_1(A) + \mu_2(A) \text{ for all } A \in \mathscr{A}$$

is also a measure.

Carathéodory Extension Theorem

In very few examples (like the Dirac measure) it is possible to define a measure by explicitly saying which values it assigns to *all* measurable sets. The following theorem shows that it is not necessary to do this and one can define the measure only on a smaller collection of sets.

Theorem 3.1.1. *[Carathéodory Extension Theorem] Let $\mathscr{A}$ be an* algebra *of subsets of X. If $\mu^* : \mathscr{A} \to \mathbb{R}^+$ satisfies*

(i) $\mu^*(\emptyset) = 0$; $\mu^*(X) < \infty$;

(ii) If $\{A_n, \quad n \in \mathbb{N}\} \subset \mathscr{A}$ is a countable collection of pairwise disjoiint sets in the algebra $\mathscr{A}$ and

$$\bigcup_{n=1}^{\infty} A_n \in \mathscr{A} \qquad \Rightarrow \qquad \mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu^*(A_n).$$

Then there exists a unique *measure $\mu : \mathscr{B}(\mathscr{A}) \to \mathbb{R}^+$ on the $\sigma-$algebra $\mathscr{B}(\mathscr{A})$ generated by $\mathscr{A}$ which extends μ^* (in the sense that it has the prescribed values on the sets of $\mathscr{A}$). We will refer to μ^* as a premeasure.*

Remark that since $\mathscr{A}$ is only an algebra and not a $\sigma-$algebra, $\bigcup_{n=1}^{\infty} A_n \in \mathscr{A}$ does not have to belong to $\mathscr{A}$. Thus, (ii) has to hold only for collections of sets A_n for which the countable union happens to still belong to $\mathscr{A}$. Thus, the theorem states that if we have a function μ^* that *behaves like* a measure on an algebra, it can indeed be extended (and uniquely) to a measure.

This theorem will be used mostly in the following two ways:

1. To define a measure on the $\sigma-$algebra $\mathscr{B}(\mathcal{S})$ generated by $\mathcal{S}$, it is enough to define μ on $\mathcal{S}$ in such a way that it satisfies the assumptions of the Theorem on the *algebra* generated by $\mathcal{S}$. This automatically defines a measure on the whole $\sigma-$algebra $\mathscr{B}(\mathcal{S})$.

2. If we have two measures μ, ν and we want to show that $\mu = \nu$, it is enough to check that $\mu(A) = \nu(A)$ for all $A \in \mathscr{A}$ where $\mathscr{A}$ is an algebra that generates the $\sigma-$algebra of all measurable sets. Then, by the uniqueness part of the Theorem, the measures μ and ν are the same measure.

We can now formally define the following measures

Example 3.1.6. Let $a, b \in \mathbb{R}$ with $a < b$ consider the interval $[a, b] \subset \mathbb{R}$. We can define Lebesgue meausre λ on intervals $(c, d) \subset [a, b]$, by setting $\lambda((c, d)) = d - c$. This clearly defines by additivity also a premeasure λ on the algebra consisting of finite unions of intervals. If $A = \cup_{i=1}^{n}(a_i, b_i)$ and (a_i, b_i) are *disjoint* intervals, we just define

$$\mu(A) = \sum_{i=1}^{n} b_i - a_i.$$

Since the condition (ii) of the theorem holds, this automatically defines the Lebesgue measure on the $\sigma-$algebra generated by all intervals, that is on all Borel subsets of $[a, b]$. The same method works to define Lebesgue measure on the whole of $\mathbb{R}$ however as stated the Carathéodory Extension Theorem only holds when the premeasure is finite. However in fact it holds with a slightly weaker assumption (σ-finiteness) which allows us to define Lebesgue measure on the whole of $\mathbb{R}$, see remark 3.1.2.

Example 3.1.7. Let $X = \mathbb{T}^2$. Consider sets of the form $[a, b] \times [c, d]$, that we call rectangles. Define a measure λ by setting

$$\lambda([a, b] \times [c, d]) = (b - a)(d - c).$$

The collection of all finite unions of rectangles is an algebra. Extending the definition of λ to union of rectangles by additivity, condition (ii) of the theorem automatically holds. Thus, the Theorem guarantees that we defined a Lebesgue measure on the $\sigma-$algebra generated by all rectangles, which coincides with all Borel subsets. This is again the $2-$dimensional Lebesgue measure λ on $\mathbb{T}^2$.

Example 3.1.8. If we have a non-negative Riemann integrable (shortly we will extend this to Lebesgue integrable) function $f : \mathbb{R} \to \mathbb{R}^+$ then for any subinterval $A \subset \mathbb{R}$ we can define

$$\mu_f(A) = \int_A f(x)\mathrm{dx}.$$

Now if we consider a disjoint finite union of subintervals $A_1, \ldots, A_n \subset \mathbb{R}$ we can write

$$\mu_f\left(\cup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu_f(A_i).$$

Now any finite union of subintervals can be rewritten as a disjoint finite union of subintervals, the set of finite unions of subintervals forms an algebra and μ_f satisfies the conditions to apply Thoerem 3.1.1. Thus we can extend μ_f to a measure on $(\mathbb{R}, \mathscr{B})$, since the σ-algebra generated by our algebra is the Borel σ-algebra.

Extras: Remarks

Remark 3.1.1. Condition (iii) in the definition of algebra, that is

(iii) $\mathscr{A}$ is closed under finite unions, that is if $A_1, \ldots, A_n \in \mathscr{A}$, then

$$\bigcup_{i=1}^{n} A_i \in \mathscr{A};$$

can be equivalently replaced by the following condition

(iii)' $\mathscr{A}$ is closed under intersections, that is if $A, B \in \mathscr{A}$, then $A \cap B \in \mathscr{A}$;

In some books, the definition of algebra is given using $(i), (ii), (iii)'$.

Exercise 3.1.2. Show that a set satisfies conditions $(i), (ii), (iii)$ if and only if it satisfies $(i), (ii), (iii)'$.

Remark 3.1.2. The condition $\mu^*(X) < \infty$ in the Extension theorem can be relaxed. It is enough that $X = \cup_n X_n$ where each X_n is such that $\mu^*(X_n) < \infty$. We say in this case that the resulting measure μ is $\sigma-$finite. For example, the Lebesgue measure on $\mathbb{R}$ is $\sigma-$finite since

$$\mathbb{R} = \cup_n [-n, n], \quad \text{and} \quad \lambda([-n, n]) = 2n < \infty.$$

3.2 Measure preserving transformations

In this section we present the definition and many examples of *measure-preserving transformations*. Let $(X, \mathscr{B}, \mu)$ be a measure space. For the ergodic theory part of our course, we will use the notation $T : X \to X$ for the map giving a discrete dynamical system, instead than $f : X \to X$ (T stands for transformation). This is because we will use the letter f for functions $f : X \to \mathbb{R}$ (which will play the role of observables).

Definition 3.2.1. A transformation $T : X \to X$ is *measurable*, if for any measurable set $A \in \mathscr{B}$ the preimage is again measurable, that is $T^{-1}(A) \in \mathscr{B}$.

One can show that if (X, d) is a metric space, $\mathscr{B} = \mathscr{B}(X)$ is the Borel $\sigma-$algebra and $T : X \to X$ is continuous, than in particular T is measurable. All the transformations we will consider will be measurable.

[Even if not explicitly stated, when in the context of ergodic theory we consider a transformation $T : X \to X$ on a measurable space $(X, \mathscr{B})$ we implicitly assume that it is measurable.]

Definition 3.2.2. A transformation $T : X \to X$ is *measure-preserving* if it is measurable and if for all measurable sets

$$\mu(T^{-1}(A)) = \mu(A), \qquad \text{for all } A \in \mathscr{B}. \tag{3.4}$$

We also say that the transformation T *preserves* μ.

If μ satisfies (3.4), we say that the measure μ is *invariant* under the transformation T.

Notice that in (3.4) one uses T^{-1} and not T. This is essential if T is not invertible, as it can be seen in Example 3.2.1 below (on the other hand, one could alternatively use forward images if T is invertible, see Remark 3.2.2 below). Notice also that we need to assume that T is measurable to guarantee that $T^{-1}(A)$ is measurable, so that we can consider $\mu(T^{-1}(A))$ (recall that a measure is defined only on measurable sets).

We will see many examples of measure-preserving transformations both in this lecture and in the next ones.

Remark 3.2.1. Let T be measurable. Let us define $T_*\mu : \mathscr{B} \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$T_*\mu(A) = \mu(T^{-1}(A)), \qquad A \in \mathscr{B}.$$

One can check that $T_*\mu$ is a measure. The measure $T_*\mu$ is called *push-forward* of μ with respect to T. Equivalently, T is measure-preserving if and only if $T_*\mu = \mu$.

Exercise 3.2.1. Verify that if μ is a measure on the measurable space $(X, \mathscr{B})$ and T is a measurable transformation, the push-forward $T_*\mu$ is a measure on $(X, \mathscr{B})$.

Thanks to the extension theorem, to prove that a measure is invariant, it is not necessary to check the measure-preserving relation (3.4) for *all* measurable sets $A \in \mathscr{B}$, but it is enough to check it for a smaller class of subsets:

Lemma 3.2.1. *If the $\sigma-$algebra $\mathscr{B}$ is generated by an algebra $\mathscr{A}$ (that is, $\mathscr{B} = \mathscr{B}(\mathscr{A})$), then μ is preserved by T if and only if*

$$\mu(T^{-1}(A)) = \mu(A), \qquad \textit{for all } A \in \mathscr{A}, \tag{3.5}$$

that is, it is enough to check the measure preserving relation for the elements on the generating algebra $\mathscr{A}$ and then it automatically holds for all elements of $\mathscr{B}(\mathscr{A})$.

Proof. Consider the two measures μ and $T_*\mu$. If (3.5) holds, then μ and $T_*\mu$ are equal on the algebra $\mathscr{A}$. Moreover, both of them satisfy the assumptions of the Extension theorem, since they are measures. The uniqueness part of the Extension theorem states that there is a *unique* measure that extends their common values on the algebra. Thus, since μ and $T_*\mu$ are both measures that extend the same values on the algebra, by uniqueness they must coincide. Thus, $\mu = T_*\mu$, which means that T is measure-preserving. The converse is trivial: if μ and $T_*\mu$ are equal on elements of $\mathscr{B}(\mathscr{A})$, in particular they coincide on $\mathscr{A}$. □

As a consequence of this Lemma, to check that a transformation T is measure preserving, it is enough to check it for:

($\mathbb{R}$) *intervals* $[a, b]$ if $X = \mathbb{R}$ or $X = I \subset \mathbb{R}$ is an interval and $\mathscr{B}$ is the the Borel $\sigma-$algebra;

($\mathbb{R}^2$) *rectangles* $[a, b] \times [a, b]$ if $X = \mathbb{R}^2$ or $X = [0, 1]^2$ and $\mathscr{B}$ is the the Borel $\sigma-$algebra;

(S^1) *arcs* if $X = S^1$ with the Borel $\sigma-$algebra;

(Σ) *cylinders* $C_{-m,n}(a_{-m}, \dots, a_n)$ if X is a shift space $X = \Sigma_N$ or $X = \Sigma_A$ and $\mathscr{B}$ is the $\sigma-$algebra;

[This is because *finite unions* of the subsets above mentioned (intervals, rectangles, arcs, cylinders) form algebras of subsets. If one checks that $\mu = T_*\mu$ on these subsets, by additivity of a measure they coincide on the whole algebra of their finite unions. Thus, by the Lemma, μ and $T_*\mu$ coincide on the whole $\sigma-$algebra generated by them, which is in all cases the corresponding Borel $\sigma-$algebra.]

3.2.1 Examples of measure-preserving transformations

Let us now give several examples of measure-preserving transformations. These are all transformations that we have encountered in the previous chapters, but we now show that they indeed preserves natural measures.

Example 3.2.1. [**Doubling map**] Consider $(X, \mathscr{B}, \lambda)$ where $X = [0, 1]$ and λ is the Lebesgue measure on the Borel $\sigma-$algebra $\mathscr{B}$ of X. Let $f(x) = 2x \mod 1$ be the doubling map. Let us check that f preserves λ. Since

$$f^{-1}[a, b] = \left[\frac{a}{2}, \frac{b}{2}\right] \cup \left[\frac{a+1}{2}, \frac{b+1}{2}\right],$$

we have

$$\lambda(f^{-1}[a, b]) = \frac{b-a}{2} + \frac{(b+1)-(a+1)}{2} = b - a = \lambda([a, b]),$$

so the relation (3.4) holds for all intervals. Since if $I = \cup_i I_i$ is a (finite or countable) union of *disjoint* intervals $I_i = [a_i, b_i]$, we have

$$\lambda(I) = \sum_i |b_i - a_i|,$$

one can check that $\lambda(f^{-1}(I)) = \lambda(I)$ holds also for all I which belong to the algebra of finite unions of intervals. Thus, by the extension theorem (see Lemma 3.2.1 and (S^1)), we have $\lambda(f^{-1}(B)) = \lambda(B)$ for all Borel measurable sets.

On the other hand check that $\lambda(f([a, b])) = 2\lambda([a, b])$, so $\lambda(f([a, b])) \neq \lambda([a, b])$. This shows the importance of using T^{-1} and not T in the definition of measure preserving.

Example 3.2.2. [**Rotations**] Let $R_\alpha : S^1 \to S^1$ be a rotation. Let λ be the Lebesgue measure on the circle, which is the same than the $1-$dimensional Lebesgue measure on $[0,1]$ under the identification of S^1 with $[0,1]/\sim$. The measure $\lambda(A)$ of an arc is then given by the arc length divided by 2π, so that $\lambda(S^1) = 1$.

Remark that if R_α is the counterclockwise rotation by $2\pi\alpha$, than $R_\alpha^{-1} = R_{-\alpha}$ is the clockwise rotation by $2\pi\alpha$. If A is an arc, it is clear that the image of the arc under the rotation has the same arc length, so

$$\lambda(R_\alpha^{-1}(A)) = \lambda(A), \qquad \text{for all arcs } \ A \subset S^1.$$

Thus, by the Extension theorem (see (S^1) above), we have $(R_\alpha)_*\lambda = \lambda$, that is R_α is measure preserving.

In this Example, one can see that we also have $\lambda(R_\alpha(A)) = \lambda(R_\alpha^{-1}(A)) = \lambda(A)$. This is the case more in general for *invertible* transformations:

Remark 3.2.2. Suppose T is invertible with T^{-1} measurable. Then T preserves μ if and only if

$$\mu(TA) = \mu(A), \qquad \text{for all measurable sets } A \in \mathscr{B}. \tag{3.6}$$

Exercise 3.2.2. Prove the remark, by first showing that if T is invertible (injective and surjective) one has

$$T(T^{-1}(A)) = A, \qquad T^{-1}(T(A)) = A.$$

[Notice that this is false in general if T is not invertible. For any map T one has the inclusions

$$T(T^{-1}(A)) \subset A, \qquad A \subset T^{-1}(T(A)),$$

but you can give examples where the first inclusion can be strict if T is not surjective and the second inclusion $A \subset T^{-1}(T(A))$ is strict if T is not injective.]

In the next example, we will use the following:

Remark 3.2.3. Let $(X, \mathscr{B}, \mu)$ be a measure-space. If $T : X \to X$ and $S : X \to X$ both preserve the measure μ, than also their composition $T \circ S$ preserves the measure μ. Indeed, for each $A \in \mathscr{B}$, since $T^{-1}(A) \in \mathscr{B}$ since T is measurable. Then, using first that S is measure preserving and then that T is also measure preserving, we get

$$\mu(S^{-1}(T^{-1}(A))) = \mu(T^{-1}(A)) = \mu(A).$$

Thus, $T \circ S$ is measure-preserving.

Example 3.2.3. [**Toral automorphisms**] Let $f_A : \mathbb{T}^2 \to \mathbb{T}^2$ be a toral automorphism; A denotes the corresponding invertible integer matrix. Let us show that f_A preserves the $2-$dimensional Lebesgue measure λ on the torus. As usual be identify $\mathbb{T}^2$ with the unit square $[0,1)^2$ with oposite sides identified. Since the set of all finite unions of rectangles in $[0,1)^2$ forms an algebra which generates the Borel σ-algebra of the metric space $(\mathbb{T}^2, d)$, and since $f_A^{-1} = f_{A^{-1}}$ is measurable, it is sufficient to prove $\lambda(f_A(R)) = \lambda(R)$ for all rectangles $R \subset [0,1)^2$. The image of R under the linear transformation A is the parallelogram AR. Since $|\det(A)| = 1$, AR has the same area as A. The parallelogram AR can be partitioned into finitely many disjoint polygons P_j, such that for each j we find an integer vector $\mathbf{m_j} \in \mathbb{Z}^2$ with $P_j + \mathbf{m_j} \in [0,1)^2$. Thus

$$f_A(R) = \bigcup_j (P_j + \mathbf{m_j}).$$

Since f_A is invertible, the sets $P_j + \mathbf{m_j}$ are pairwise disjoint, and hence

$$\lambda(f_A(R)) = \sum_j \lambda(P_j + \mathbf{m_j}) = \sum_j \lambda(P_j) = \lambda(R)$$

which completes the proof. (In the second equality above we have used that translations preserve the Lebesgue measure λ.)

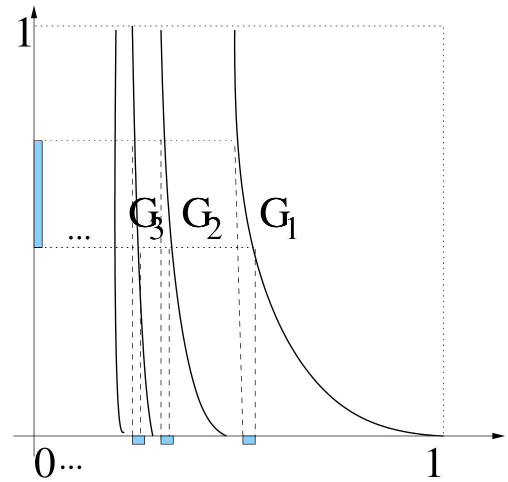


Figure 3.1: The first branches of the graph of the Gauss map.

Example 3.2.4. [**Gauss map**] Let $X = [0,1]$ with the Borel $\sigma-$algebra and let $G : X \to X$ be the Gauss map (see Figure 3.1). Recall that $G(0) = 0$ and if $0 < x \leq 1$ we have

$$G(x) = \left\{ \frac{1}{x} \right\} = \frac{1}{x} - n \qquad \text{if} \quad x \in P_n = \left(\frac{1}{n+1}, \frac{1}{n} \right].$$

The *Gauss measure* μ is the measure defined by the density $\frac{1}{(1+x)\log 2}$, that is the measure that assigns to any interval $[a,b] \subset [0,1]$ the value

$$\mu([a,b]) = \frac{1}{\log 2} \int_a^b \frac{1}{1+x} \mathrm{d}x.$$

By the Extension theorem, this defines a measure on all Borel sets. Since

$$\int_0^1 \frac{1}{1+x} = \log(1+x)|_0^1 = \log 2 - \log 1 = \log 2,$$

the factor $\log 2$ in the density is such that $\mu([0,1]) = 1$, so the Gauss measure is a probability measure.
[The Gauss measure was discovered by Gauss who found that the correct density to consider to get invariance was indeed $1/(1+x)$.]

Proposition. *The Gauss map G preserves the Gauss measure μ, that is $G_*\mu = \mu$.*

Proof. Consider first an interval $[a,b] \subset [0,1]$. Let us call G_n the branch of G which is given by restricting G to the interval P_n. Since each G_n is surjective and monotone, the preimage $G^{-1}([a,b])$ consists of countably many intervals, each of the form $G_n^{-1}([a,b])$ (see Figure 3.1). Let us compute $G_n^{-1}([a,b])$:

$$\begin{aligned} G_n^{-1}([a,b]) = \{x \text{ s.t. } G_n(x) \in [a,b]\} &= \left\{ x \text{ s.t. } a \leq \frac{1}{x} - n \leq b \right\} \\ &= \left\{ x \text{ s.t. } \frac{1}{b+n} \leq x \leq \frac{1}{a+n} \right\} = \left[\frac{1}{b+n}, \frac{1}{a+n} \right]. \end{aligned}$$

Remark also that $G_n^{-1}([a,b])$ are clearly all disjoint. Thus, by countably additivity of a measure, we have

$$\begin{aligned}
\mu(G^{-1}([a,b])) &= \mu\left(\bigcup_{n=1}^{\infty} G_n^{-1}([a,b])\right) = \mu\left(\bigcup_{n=1}^{\infty}\left[\frac{1}{b+n},\frac{1}{a+n}\right]\right) = \sum_{n=1}^{\infty}\mu\left(\left[\frac{1}{b+n},\frac{1}{a+n}\right]\right) \\
&= \sum_{n=1}^{\infty}\int_{\frac{1}{b+n}}^{\frac{1}{a+n}} \frac{1}{\log 2}\frac{\mathrm{d}x}{(1+x)} = \frac{1}{\log 2}\sum_{n=1}^{\infty}\left(\log\left(1+\frac{1}{a+n}\right) - \log\left(1+\frac{1}{b+n}\right)\right) \\
&= \frac{1}{\log 2}\sum_{n=1}^{\infty}\left(\log\left(\frac{1+a+n}{a+n}\right) - \log\left(\frac{1+b+n}{b+n}\right)\right).
\end{aligned}$$

By definition, the sum of the series is the limit of its partial sums and we have that

$$\sum_{n=1}^{N}\log\left(\frac{1+a+n}{a+n}\right) = \sum_{n=1}^{N}\log(1+a+n) - \log(a+n).$$

Remark that the sum is a telescopic sum in which consecutive terms cancel each other (write a few to be convinced), so that

$$\sum_{n=1}^{N}(\log(1+a+n) - \log(a+n)) = -\log(a+1) + \log(1+a+N).$$

Similarly,

$$\sum_{n=1}^{N}(\log(1+b+n) - \log(b+n)) = -\log(b+1) + \log(1+b+N).$$

Thus, going back to the main computation:

$$\begin{aligned}
G^{-1}([a,b]) &= \frac{1}{\log 2}\lim_{N\to\infty}\sum_{n=1}^{N}\left(\log\left(\frac{1+a+n}{a+n}\right) - \log\left(\frac{1+b+n}{b+n}\right)\right) \\
&= \frac{1}{\log 2}\lim_{N\to\infty}(\log(1+a+N) - \log(a+1) - (\log(1+b+N) - \log(b+1))) \\
&= \frac{1}{\log 2}\left[\log(b+1) - \log(a+1) + \lim_{N\to\infty}\left(\log\frac{1+a+N}{1+b+N}\right)\right] \\
&= \frac{1}{\log 2}(\log(b+1) - \log(a+1) + 0) = \frac{1}{\log 2}\int_a^b \frac{\mathrm{d}x}{(1+x)}.
\end{aligned}$$

This shows that $\mu(A) = G_*\mu(A)$ for all A intervals. By additivity, $\mu(A) = G_*\mu(A)$ on the algebra of finite unions of intervals. Thus, by the Extension theorem (see Lemma 3.2.1), $\mu = G_*\mu$. □

3.2.2 An equivalent definition of measure-preserving.

Let $(X, \mathscr{A}, \mu)$ be a *finite* measure space. Recall that a transformation $T : X \to X$ is measure preserving iff it is measurable and

$$\mu(T^{-1}(A)) = \mu(A), \qquad \text{for all } A \in \mathscr{A}.$$

An equivalent and very useful characterization of measure-preserving transformations can be given using integrals with respect to a measure (see the Appendix for a quick review of how to define integration in a measure space).

Lemma 3.2.2 (Measure-preserving via integration). *A measurable transformation $T : X \to X$ is* measure-preserving *if and only if, for any integrable function $f : X \to \mathbb{R}$ we have*

$$\int f \,\mathrm{d}\mu = \int f \circ T \,\mathrm{d}\mu. \tag{3.7}$$

[You might have seen a special case of the previous formula: if $X = \mathbb{R}^2$, λ is Lebesgue, f is Riemann integrable and T is an *area-preserving* transformation, which is equivalent to $|det(DT)| = 1$, then

$$\int f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int f \circ T(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

holds simply by the change of variables formula.]

Remark 3.2.4. One can show that in Lemma 3.2.2 rather than considering all functions which are integrable, it is enough to check that (3.7) holds for all square-integrable functions $f \in L^2(\mu)$.

Proof of Lemma 3.2.2. Let us first assume that (3.7) hold and show that T is measure preserving. Let $A \in \mathscr{A}$. Let us first remark that

$$\chi_A \circ T = \chi_{T^{-1}(A)}, \tag{3.8}$$

since by definition of characteristic function, for each $x \in X$

$$\chi_A \circ T(x) = \chi_A(T(x)) = \begin{cases} 1 & \text{if } T(x) \in A \quad \Leftrightarrow x \in T^{-1}(A) \\ 0 & \text{if } T(x) \notin A \quad \Leftrightarrow x \notin T^{-1}(A), \end{cases}$$

which coincides exactly with the definition of

$$\chi_{T^{-1}(A)} = \begin{cases} 1 & \text{if } x \in T^{-1}(A) \\ 0 & \text{if } x \notin T^{-1}(A). \end{cases}$$

Consider the function χ_A, which, as we saw in Example **??**, is measurable. Then, we have

$$\begin{aligned} \mu(T^{-1}(A)) &= \int \chi_{T^{-1}(A)}\,\mathrm{d}\mu \qquad \text{(by Step 0 in the definition of integrals)} \\ &= \int \chi_A \circ T\,\mathrm{d}\mu \qquad \text{by (3.8)} \\ &= \int \chi_A\,\mathrm{d}\mu \qquad \text{by (3.7)} \\ &= \mu(A) \qquad \text{(again by Step 0 in the definition of integrals)}\ . \end{aligned}$$

Since this can be done for any $A \in \mathscr{A}$, T is measure-preserving.

Conversely, let us assume that T is measure-preserving and let us show that (3.7) holds for any measurable function f. Let us go through the same steps that we followed in the definition of integrals:

1. Consider first the case where $f = \chi_A$ is the indicatrix of $A \in \mathscr{A}$. Then, by definition of measure-preserving and definition of integrals (Step 0), using again that $\chi_A \circ T = \chi_{T^{-1}(A)}$ in (3.8), we have

   $$\int \chi_A \circ T\,\mathrm{d}\mu = \int \chi_{T^{-1}(A)}\,\mathrm{d}\mu = \mu(T^{-1}(A)) = \mu(A) = \int \chi_A\,\mathrm{d}\mu.$$

   Thus, we showed that (3.7) holds for any characteristic function of measurable set.

2. Since integrals are *linear*, that is

   $$\int (a_1 f_1 + a_2 f_2)\mathrm{d}\mu = a_1 \int f_1 \mathrm{d}\mu + a_2 \int f_2 \mathrm{d}\mu,$$

   (3.7) holds for any simple function (see Exercise 3.2.5).

3. Consider now any non-negative measurable function $f : X \to \mathbb{R}$. If $f_n \nearrow f$ is a sequence of simple functions approximating f, one can check (see Exercise 3.2.6) that $f_n \circ T \nearrow f \circ T$ is a sequence of simple functions approximating $f \circ T$. Thus, by Step $(2)'$ of the definition of integrals (see Remark **??**), since (3.7) holds for any of the simple functions f_n, we have
$$\int f \, \mathrm{d}\mu = \lim_{n\to\infty} \int f_n \, \mathrm{d}\mu = \lim_{n\to\infty} \int f_n \circ T \, \mathrm{d}\mu = \int f \circ T \, \mathrm{d}\mu.$$
Thus, (3.7) holds for any non-negative measurable function.

4. If f is any integrable function, (3.7) holds by taking non-negative and negative parts (see Step (3) of the definition of integral) and again using linearity of integrals.

This concludes the proof. □

Example 3.2.5. Verify that if (3.7) holds for any characteristic function of measurable set then it holds for any simple function.

Example 3.2.6. Verify that if g is a simple function, then also $g \circ T$ is a simple function. Verify that if $f_n \nearrow f$, then $f_n \circ T \nearrow f \circ T$.

3.3 Poincaré Recurrence

Let $(X, \mathscr{B})$ be a measurable space and let $T : X \to X$ be a measurable transformation. Let us say that T *has a finite invariant measure* if there is a measure μ invariant under T with $\mu(X) < \infty$ (we saw many examples in the previous lecture). Just possessing a finite invariant measure has already very important dynamical consequences. We will see in this class Poincaré Recurrence and, in §3.7, the Birkhoff Ergodic Theorem. Both assume only that there is a finite measure preserved by the transformation $T : X \to X$.

Notation 3.3.1. *If $(X, \mathscr{B}, \mu)$ is a measure space, we say that a property hold for μ-almost every point and write* for $\mu - a.e$ point *if the set of $x \in X$ for which it fails has measure zero. Similarly, if $B \subset X$ is a subset, we say that property hold for μ-almost every point $x \in B$ if the set of points in B for which it fails has measure zero.*

If μ is the Lebesgue measure or if the measure is clear from the context and there is no-ambiguity, we will simply say that the property holds for almost every *point and write that it holds for* a.e. $x \in X$.

Definition 3.3.1. Let $B \subset X$ be a subset. We say that a point $x \in B$ *returns* to B if there exists $k \geq 1$ such that $T^k(x) \in B$. We say that $x \in B$ is *infinitely recurrent* with respect to B if it returns infinitely often to B, that is there exists an increasing sequence $(k_n)_{n\in\mathbb{N}}$ such that $T^{k_n}(x) \in B$.

Theorem 3.3.1 (Poincaré Recurrence, weak form). *If $(X, \mathscr{B}, \mu)$ is a measure space, T preserves μ and μ if* finite*, then for any $B \in \mathscr{B}$ with positive measure $\mu(B) > 0$, $\mu-$almost every point $x \in B$ returns to B (that is, the set of points $x \in B$ that never returns to B has measure zero).*

Before giving the formal proof, let us explain the idea behind it: if B is a set with positive measure, let us consider the preimages $T^{-n}(B)$, $n \in \mathbb{N}$. Since T is measure preserving, all the preimages have the same measure. Since the total measure of the space is finite, the sets $B, T^{-1}(B), \dots, T^{-n}(B), \dots$ cannot be all disjoint, since otherwise the measure of their union would have infinite measure. Thus, they have to intersect. Intersections give points in B which return to B (if $x \in T^{-n}(B) \cap T^{-m}(B)$ where $m > n$, then $T^n(x) \in B$ and $T^{m-n}(T^n(x)) = T^m(x) \in B$, so $T^n(x)$ returns to B). This only shows so far that *there exists* a point in B that returns. The proof strengthen this result to almost every point.

Proof of Theorem 3.3.1. Consider the set $A \subset B$ of points $x \in B$ which *do not* return to B. Equivalently, we have to prove that $\mu(A) = 0$. Consider the preimages $\{T^{-n}(A)\}_{n\in\mathbb{N}}$. Clearly, $T^{-n}(A) \subset X$, so

$$\left(\bigcup_{n\in\mathbb{N}} T^{-n}(A)\right) \subset X \qquad \Rightarrow \qquad \mu\left(\bigcup_{n\in\mathbb{N}} T^{-n}(A)\right) \le \mu(X) < \infty,$$

where we used that if $E \subset F$ are measurable sets, then $\mu(E) \le \mu(F)$ (this property of a measure, which is very intuitive, can be formally derived from the definition of measure, see Exercise 3.3.1 below). Let us show that $\{T^{-n}(A)\}_{n\in\mathbb{N}}$ are pairwise *disjoint.* If not, there exists $n, m \in N$, with $n \ne m$, such that

$$T^{-n}(A) \cap T^{-m}(A) \ne \emptyset \quad \Leftrightarrow \quad \text{there exists } x \in T^{-n}A \cap T^{-m}(A).$$

Assume that $m > n$. Then

$$T^n(x) \in A, \quad \text{and} \quad T^{m-n}(T^n x) = T^m(x) \in A,$$

but this contradicts the definition of A (all points of A do *not* return to A). Thus, $\{T^{-n}(A)\}_{n\in\mathbb{N}}$ are all disjoint. By the countable additivity property of a measure,

$$\sum_{n=1}^{\infty} \mu(T^{-n}A) = \mu\left(\bigcup_{n\in\mathbb{N}} T^{-n}(A)\right) \le \mu(X).$$

Remark now that since T is measure preserving, $\mu(T^{-n}(A)) = \mu(A)$ for all $n \in \mathbb{N}$. Thus, we have a finite series whose terms are all equal. If $\mu(A) > 0$, this cannot happen (the series with constant terms equal to $\mu(A) > 0$ diverges), so $\mu(A) = 0$ as desired. □

In the proof we used the following property of a measure, which you can derive from the properties in the definition of a measure:

Exercise 3.3.1. Let μ be a measure on $(X, \mathscr{B})$. Show using the property of a measure that if $E, F \in \mathscr{B}$ and $E \subset F$, then $\mu(E) \le \mu(F)$.

If $E \subset F$ is a strict inclusion, does it imply that $\mu(E) < \mu(F)$? Justify.

We can prove actually more and show that almost every point is *infinitely recurrent*:

Theorem 3.3.2 (Poincaré Recurrence, strong form). *If $(X, \mathscr{B}, \mu)$ is a measure space, T preserves μ and μ if* finite, *then for any $B \in \mathscr{B}$ with positive measure $\mu(B) > 0$, $\mu-$almost every point $x \in B$ is* infinitely recurrent *to B (that is, the set of points $x \in B$ that returns to B only finitely many times has measure zero).*

Proof. Let A be the set of points in B that *do not* return to B infinitely many times. We want to prove that $\mu(A) = 0$. The points in A are the points which return only finitely many (possibly zero) times. Thus, if $x \in A$, for all n sufficiently large $T^n(x)$ is outside B, that is

$$A = \{x \in B \ \text{ such that there exists } k \ge 1 \text{ such that } T^n(x) \notin B \text{ for all } n \ge k\}.$$

Consider the set

$$A_0 = \{x \in B \ \text{ such that } T^n(x) \notin B \text{ for all } \ n > 0 \ \},$$

and the sets $A_k = T^{-k}(A_0)$ for $k \ge 1$. Then, if $x \in T^{-k}(A_0)$, $T^k(x) \in A_0$, so $T^k(x) \in B$ and $T^n(T^k(x)) = T^{k+n}(x) \notin B$ for all $n > 0$. Thus

$$A_k = \{x \ \text{ such that } T^k(x) \in B \text{ and } T^n(x) \notin B \text{ for all } \ n > k \ \}.$$

One can then write

$$A = B \cap \bigcup_{k=0}^{+\infty} A_k. \tag{3.9}$$

[Indeed, if $x \in A$, let k be the largest integer such that $T^k(x) \in B$, which is well defined since there are only finitely many such integers by definition of A. Then $x \in A_k$. Conversely, if $x \in B \cap A_k$ for some k, $T^n(x) \notin B$ for all $n > k$, so x can return only finitely many times and belongs to A.]

From (3.9) and Exercise (3.3.1), we have

$$A \subset \bigcup_{k=0}^{+\infty} A_k \qquad \Rightarrow \qquad \mu(A) \leq \mu\left(\bigcup_{k=0}^{+\infty} A_k\right).$$

Thus, to prove that $\mu(A) = 0$ it is enough to prove that the union $\cup_k A_k$ has measure zero.

Reasoning as in the proof of the weak version of Poincaré Recurrence, since we showed that $\{A_k\}_{k\in\mathbb{N}}$ are pairwise disjoint, we have

$$\sum_{k=0}^{\infty} \mu(A_k) = \mu\left(\bigcup_{k=0}^{+\infty} A_k\right) \leq \mu(X) < +\infty.$$

Furthermore, $\mu(A_k) = \mu(A_0)$ for all $k \in \mathbb{N}$ since $A_k = T^{-k}(A_0)$ and T is measure preserving. Thus, the only possibility to have a convergent series with non-negative equal terms is $\mu(A_0) = \mu(A_k) = 0$ for all $k \in \mathbb{N}$. But then

$$\sum_{k=0}^{\infty} \mu(A_k) = 0 \qquad \Rightarrow \qquad \mu(A) \leq \mu\left(\bigcup_{k=0}^{+\infty} A_k\right) = \sum_{k=0}^{\infty} \mu(A_k) = 0,$$

so $\mu(A) = 0$ and almost every point in B is infinitely recurrent. □

Remarks

1. Notice that recurrence is different than density. If $x \in B$ is periodic of period n, for example, it *does* return to B infinitely often even if its orbit is *not* dense, since $T^{kn}(x) = x \in B$ for any $k \in \mathbb{N}$. For example, consider a rational rotation R_α where $\alpha = p/q$. Then all orbits are periodic, so no points is dense. On the other hand, R_α preserves the Lebesgue measure and the conclusion of Poincaré Recurrence Theorem holds. Given any measurable set B, any point of B is infinitely recurrent.

2. If μ is *not* finite, Poincaré Recurrence Theorem does not hold. Consider for example $X = \mathbb{R}$ with the Borel $\sigma-$algebra and the Lebesgue measure λ. Let $T(x) = x + 1$ be the translation by 1. Then T preserves λ, but no point $x \in \mathbb{R}$ is recurrent: all points tend to infinity under iterates of T.

Exercise 3.3.2. (a) Let $X = \mathbb{R}^2$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation

$$T(x, y) = (x + y, y) \qquad \text{given by the matrix} \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Show that the conclusion of Poincaré Recurrence Theorem fails for T.

(b) Let $X = \mathbb{T}^2$ and let $T : \mathbb{T}^2 \to \mathbb{T}^2$ be the toral automorphism given by A, that is

$$T(x, y) = (x + y \mod 1,\ y \mod 1).$$

Show that in this case, for any rectangle $R = [a, b] \times [c, d] \subset \mathbb{T}^2$ *all* points $(x, y) \in R$ are infinitely recurrent to R.

[*Hint*: separate the two cases y rational and y irrational.]

Extra 1: Poincaré Recurrence for incompressible transformations

Poincaré Recurrence holds not only for measure-preserving transformations, but more in general for a larger class of transformations called *incompressible*. In Exercise 3.3.3 we outline the steps of an alternative proof of the strong form of Poincaré Recurrence which holds also for incompressible transformations.

Definition 3.3.2. Let $(X, \mathscr{B}, \mu)$ be a *finite* measure space. A transformation $T : X \to X$ is called *incompressible* if for any $A \in \mathscr{B}$

$$A \subset T^{-1}(A) \quad \Rightarrow \quad \mu(T^{-1}(A)) = \mu(A).$$

Notice that here we only assume that the inclusion $A \subset T^{-1}(A)$ holds and not that A is invariant. Clearly if a transformation is measure preserving, in particular it is also incompressible.

Theorem 3.3.3 (Poincaré Recurrence for incompressible transformations). *If $(X, \mathscr{B}, \mu)$ be a* finite *measure space and $T : X \to X$ is* incompressible*, then, for any $B \in \mathscr{B}$ with positive measure, $\mu-$almost every point $x \in B$ is* infinitely recurrent *to B.*

Exercise 3.3.3. Prove and use the following steps to give a proof of Theorem 3.3.3:

1. The set $E \subset A$ of points in $A \in \mathscr{B}$ which *are* infinitely recurrent can be written as

$$E = A \cap \bigcap_{n \in \mathbb{N}} E_n, \qquad \text{where} \quad E_n = \bigcup_{k \geq n} T^{-k} A;$$

2. The sets E_n are nested, that is $E_{n+1} \subset E_n$, and one has $\mu(\cap_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty} \mu(E_n)$;
   [*Hint*: write $\mu(E_0 \setminus \cap_{n\in\mathbb{N}} E_n)$ as a telescopic series using the disjoint sets $E_n \setminus E_{n+1}$.]
3. Show that $T^{-1}(E_n) = E_{n+1}$. Deduce that $\lim_n \mu(E_n) = \mu(E_0)$.
   Conclude by using the remark that $A \setminus \cap_{n\in\mathbb{N}} E_n \subset E_0 \setminus \cap_{n\in\mathbb{N}} E_n$.

Extra 2: Multiple recurrence and applications to arithmetic progressions.

A stronger version of Poincaré Recurrence, known as *Multiple Recurrence*, turned out to have an elegant applications to an old problem in combinatorics, the one of finding arithmetic progressions in subsets of the integer numbers.

Definition 3.3.3. An *arithmetic progression of length N* is a set of the form

$$\{a, a+b, a+2b, \ldots, a+(N-1)b\} = \{a + kb, \quad \text{where } a, b \in \mathbb{Z},\ b \neq 0,\ k = 0, \ldots, N-1\}. \tag{3.10}$$

For example, $5, 8, 11, 14, 17$ is an arithmetic progression of length 5 with $a = 5$ and $b = 3$.

Consider the set $\mathbb{Z}$ and imagine of coloring the integers with a finite number r of colors. Formally, consider a partition

$$\mathbb{Z} = B_1 \cup \ldots B_r, \qquad \text{where the sets } B_i \subset \mathbb{Z} \text{ are disjoint.} \tag{3.11}$$

(each set represents a color). Are there arbitrarily long arithmetic progressions of numbers all of the same color?

Theorem 3.3.4 (Van der Warden). *If $\{B_1, \ldots, B_r\}$ is a finite partition of $\mathbb{Z}$ as in (3.11), there exists a $1 \leq j \leq r$ such that B_j contains arithmetic progressions of arbitrary length, that is, for any N there exists $a, b \in \mathbb{Z}$, $b \neq 0$, such that $\{a + kb\}_{k=0}^{N-1} \subset B_j$.*

This Theorem can be proved using topological dynamics[5]. A proof can be found in the book by Pollicott and Yuri. A stronger result is true. The *density* (or upper density) of a subset $\mathcal{S} \subset \mathbb{Z}$ is defined as

[5] The original proof was given by Van der Waerden in 1927. The dynamical proof is due to Fursterberg and Weiss in 1978.

$$\rho(\mathcal{S}) = \limsup_{n\to\infty} \frac{Card\{\ k \in \mathcal{S}, \quad -n \le k \le n\ \}}{2n+1}.$$

Thus, we consider the proportion of integers contained in $\mathcal{S}$ in each block of the form $[-n, n] \cap \mathbb{Z}$ and take the limit (if it exists, otherwise the limsup) as n grows. A set $\mathcal{S} \subset \mathbb{Z}$ has *positive (upper) density* if $\rho(\mathcal{S}) > 0$.

Theorem 3.3.5 (Szemeredi)**.** *If $\mathcal{S} \subset \mathbb{Z}$ has* positive (upper) density*, it contains arithmetic progressions of arbitrary length, that is, for any N there exists $a, b \in \mathbb{Z}$, $b \neq 0$, such that $\{a + kb\}_{k=0}^{N-1} \subset \mathcal{S}$.*

This result was conjectured in 1936 by Erdos and Turan. The theorem was first proved by Szemeredi in 1969 for $N = 4$ and then in 1975 for any N. Szemeredi's proof is combinatorial and very complicated. A few years later, in 1977, Furstenberg gave a proof of Szemeredi's theorem by using ergodic theory. The essential ingredient in his proof, which is very elegant, is based on a stroger version of Poincaré Recurrence, known as *Multiple Recurrence*:

Theorem 3.3.6 (Multiple Recurrence)**.** *If $(X, \mathscr{B}, \mu)$ is a measure space, T preserves μ and μ if* finite*, then for any $B \in \mathscr{B}$ with positive measure $\mu(B) > 0$ and any $N \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that*

$$\mu\left(B \cap T^{-k}(B) \cap T^{-2k}(B) \cap \cdots \cap T^{-(N-1)k}(B)\right) > 0. \tag{3.12}$$

The theorem conclusion means that there exists a positive measure set of points of B which return to B *along an arithmetic progression*: if x belongs to the intersection (3.12), then $x \in B$, $T^k(x) \in B, T^{2k}(x) \in B, \ldots, T^{(N-1)k}(x) \in B$, that is, returns to B happen along an arithmetic progression of return times of length N.

From the Multiple Recurrence theorem, one can deduce Szemeredi Theorem in few lines (it is enough to set up a good space and map, which turns out to be a shift map on a shift space, find an invariant measure and translate the problem of existence of arithmetic progressions into a problem of recurrence along an arithmetic sequence of times). A reference both for this simple argument is again the book by Pollicott and Yuri, see §16.2 (in the same Chapter 16 you can find also a sketch of the full proof of the Multiple Recurrence Theorem. The proof is quite long and involved and uses more tools in ergodic theory).

A much harder question, open until recently, was whether there are arbitrarily long arithmetic progressions such that all the elements $a + kb$ are *prime numbers.* Unfortunately Szemeredi theorem does not apply if we take as set $\mathcal{S}$ the set of prime numbers: indeed, primes have zero density. Recently, Green and Tao gave a proof that the primes contain arbitrarily large arithmetic progressions. The proof is a mixture of ergodic theory and additive combinatorics.

3.4 First return times and Kac lemma

When it was first proved, Poincaré Recurrence theorem was considered for long time paradoxical, especially by physicists. We give now first give an illustration of what might seem paradoxical, then explain how this apparent paradox can be resolved by studying what are the expected return times. We will do this by introducing induced maps and first return times, as well as Rohlin towers and skyscrapers, which are fundamental constructions in ergodic theory of independent importance.

Is Poincaré Recurrence a paradox?

Let X be the phase space of a physical system, for example let X include all possible states of molecules in a box. The $\sigma-$algebra $\mathscr{B}$ represents the collections of *observable states* of the system and $\mu(A)$ is the probability of observing the state A. If T gives the discrete time evolution of the system, it is reasonable to expect that if the system is in equilibrium, T preserves μ, that is, the probability of observing a certain state is independent on time. Thus, we are in the set up of Poincaré Recurrence theorem. Consider now an initial state in which all the particles are in half of the box (for example imagine of having a wall which separates the box and then removing it). By Poincaré Recurrence Theorem, *almost surely*, all the molecules will return at some point in the same half of the box. This seems counter-intuitive. In reality, this is not a paradox, but simply the fact that the event will happen *almost surely* does not say anything about the *time* it will take to happen again (the *recurrence time*). Indeed, one can show that (if the transformation is ergodic, see next lecture) the average recurrence time is inversely proportional to the measure of the set to which one wants to return. Thus, since the phase space is huge and the observable corresponding to all molecules in half of the box has extremely small measure is this huge space, the time it will take will take to see again this configuration is also huge, probably longer than the age of the universe.

We will now make this rigorous introducing the notion of first return time.

First return times and induced transformations.

Let $(X, \mathscr{A}, \mu)$ be a probability space and let T be a measure-preserving transformation of $(X, \mathscr{A}, \mu)$. Fix a set $B \in \mathscr{A}$ with $\mu(B) > 0$. Let us define the *first return time function* r_B (of T to B) as the function $r_B : B \to \mathbb{N} \cup \{+\infty\}$ which maps $x \in B$ to $r_B(x)$ given by

$$r_B(x) = \begin{cases} \min\{n \in \mathbb{N}, n \geq 1, \text{such that } T^n(x) \in B\} \\ +\infty \quad \text{otherwise.} \end{cases}$$

Thus, $r_B(x)$, when finite, is a *return time* (of the orbit of $x \in B$ back to B), i.e. $T^{r_B(x)} \in B$, and it's the *first return time* (i.e. the smallest of all return times), so that $T^i(x) \notin B$ for all $1 \leq i \leq n-1$.

The (the weak form of) Poincaré recurrence (since $\mu-$a.e. $x \in B$ returns) guarantees that $r_B < +\infty$ for $\mu-$a.e. $x \in B$. Thus, apart from a set B_∞ with $\mu(B_\infty) = 0$, we can define the following map.

Definition 3.4.1. The *induced transformation* T_B (or first return map of T to B) is the (almost everywhere defined) map $T_B : B \to B$ given by

$$T_B(x) = T^{r_B(x)}(x).$$

Thus, the map T_B is an *acceleration* of T, obtained by applying as many iterates of T as it is necessary to come back to B.

One can define, by *restriction* to B, an *induced σ-algebra* $\mathscr{A}_B$ given by

$$\mathscr{A}_B := \{A \cap B, \qquad A \in \mathscr{A}\}$$

and an *induced measure* μ_B on $(B, \mathscr{A}_B)$ given by restriction (and rescaled to be again a probability measure), i.e.

$$\mu_B(A) = \frac{\mu(A \cap B}{\mu(B)}, \qquad \forall A \in \mathscr{A}.$$

One can show that T_B preserves the measure μ_B and hence T is a measure preserving transformation of the measure space $(B, \mathscr{A}_B, \mu_B)$.

Considering induced transfomations, or *inducing*, is a very important technique in dynamics which is often used to study dynamical systems. Often indeed, suitable chosen induced maps have better dynamical properties of the original map (e.g. properties as *bounded distorsion*). Furthermore, inducing on on smaller and smaller sets,

Rohlin towers and Kakutani skyscrapers.

The original map T can be recovered from the induced transfomation T_B through the following construction. For any $n \in \mathbb{N}$, let us define B_n to be the n^{th} level set of the return time function r_B, i.e.

$$B_n := \{x \in B, \qquad r_B(x) = n\} = \{x \in B, \quad \text{s.t. } T^n(x) \in B, \text{ but } \ T^i(x) \notin B \text{ for } 1 \leq i \leq n-1\}.$$

Then clearly $B \backslash B_\infty = \cup_{n \geq 1} B_n$, where B_∞ defined above is the set of point which don't return and have measure zero by Poincaré recurrence, so $\mu(B) = \mu\left(\cup_{n \geq 1} B_n\right)$.

Assume that B is *sweeping*, namely

$$\bigcup_{n\in\mathbb{N}} T^n(B) = X.$$

Then, up to a measure zero set, we can represent

$$X = \bigcup_{n\in\mathbb{N}} \bigcup_{i=0}^{n-i} T^i(B_n).$$

The following representation is called a *skyscrapers*. For each n, $\cup_{i=0}^{n-i} T^i(B_n)$ is a *tower* (or Rohlin tower) of the skyscrapers, and the disjoint sets $T^i(B_n)$ for $i = 0, \dots n-1$ are the *floors* of the skyscrapers.

Kac's Lemma.

We have the following result about the average return times to B:

Lemma 3.4.1 (Kac's Lemma)**.** *Let $T : (X, \mathscr{A}, \mu) \to (X, \mathscr{A}, \mu)$ where μ is a probability measure. Let $B \in \mathscr{A}$ be such that $\mu(B) > 0$. Assume in addition that*

1. *T is invertible,*
2. *B is sweeping, so $\bigcup_{n\in\mathbb{N}} T^n(B) = X$.*

Then:

$$\int_B r_B d\mu = 1.$$

As a corollary, recalling that $\mu_B = \mu/\mu(B)$, we have that

$$\int_B r_B d\mu_B = \frac{1}{\mu(B)}.$$

Thus, the *average* return time to B is $1/\mu(B)$. This explains the apparent paradox.

3.5 Invariant measures for continuous transformations

We have seen that the existence of finite invariant measures is a key property, since it already guarantees non trivial dynamical consequences like Poincaré recurrence. In this section we will show that finite invariant measures always exist when the transformation T is a continuous map of a compact metric space[6].

Throughout this section, we will assume that (X, d) is a *compact* metric space and consider it as a measurable space $(X, \mathscr{B})$ where $\mathscr{B}$ Borel σ-algebra. Let $T : X \to X$ be a topological dynamical system (recall that this means that T is continuous) and assume that T is also a measurable map of $(X, \mathscr{B})$.

Let us introduce some notation. Let $\mathcal{M}(X)$ (respectively $\mathcal{P}(X)$) be the space of *Borel finite* (resp. *probability*) invariant measures, i.e. finite (or probability) measures on $(X, \mathscr{B})$ where $\mathscr{B}$ is the Borel $\sigma-$algebra of X. Thus

$$\mathcal{M}(X) = \{\mu \text{ measure on } (X, \mathscr{B}) \text{ s.t. } \mu(X) < +\infty\};$$
$$\mathcal{P}(X) = \{\mu \text{measure on } (X, \mathscr{B}) \text{ s.t. } \mu(X) = 1\} \subset \mathcal{M}(X).$$

Remark that $\mathcal{P}(X) \subset \mathcal{M}(X)$ and given any $\mu \in \mathcal{M}(X)$, since $\mu(X) < +\infty$, we can *normalize* it by considering the rescaled measure $\mu/\mu(X)$ (i.e. the measure which gives value $\mu(A)/\mu(X)$ to $A \in \mathbb{B}$) so that $\mu/\mu(X) \in \mathcal{P}(T)$.

[6] There are other classes of transformations for which one can find invariant measures (for example piecewise expanding maps) but the dynamical tools and techniques developed to prove their existence are sometimes quite sophisticated (for example, transfer operators). Finally, there are many other dynamical systems for which proving the existence of invariant measures is a difficult or still open question.

Denote now by $\mathcal{M}_T(X)$ (resp. $\mathcal{P}_T(X)$) the measures in $\mathcal{M}(X)$ (resp. $\mathcal{P}(X)$) which are T-*invariant*:

$$\mathcal{M}_T(X) = \{\mu \in \mathcal{M}(X) \text{s.t. } T_*\mu = \mu\};$$
$$\mathcal{P}_T(X) = \{\mu \in \mathcal{P}(X) \text{s.t. } T_*\mu = \mu\}.$$

Since if $\mu \in \mathcal{M}_T(X)$, the normalized measure $\mu/\mu(X) \in \mathcal{P}_T(X)$, so that if $\mathcal{M}_T(X) \neq \emptyset$, also $\mathcal{P}_T(X) \neq \emptyset$.

The following result guarantees the existence of finite invariant measures in $\mathcal{M}_T(X)$ for a continuous T:

Theorem 3.5.1 (Krylov-Bogolyubov)**.** *For any T is a continuous map of a compact metric space X, the set $\mathcal{M}_T(X)$ (and hence also $\mathcal{P}_T(X)$) is non empty, i.e. there exists invariant (probability) Borel measures.*

The proof requires some notions of functional analysis, but contains a central idea in ergodic theory (building invariant measures as limit of measures supported on orbits). We will hence give the proof taking for granted some facts from functional analysis.

Let us first define one last symbol, $\mathcal{C}(X)$, for the space of continuous functions:

$$\mathcal{C}(X) = \{f : X \to \mathbb{R}, \quad f \text{ continuous}\}.$$

(here the choice of calling the transformation T is important, so that f denotes a function). Let us now defined a notion of *convergence for* (sequences of) *measures* (in $\mathcal{M}_T(X)$ or $\mathcal{P}_T(X)$), which is known as *weak-convergence*:

Definition 3.5.1. [Weak convergence of measures] We say that a sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_T(X)$ (or $\mathcal{P}_T(X)$) *converges* (*weak star*) to $\mu \in \mathcal{M}_T(X)$ (or in $\mathcal{P}_T(X)$) and that μ is the *weak star*) *limit* of the sequence $(\mu_n)_{n\in\mathbb{N}}$ and write

$$\mu = \lim_{n\to\infty} \mu_n, \qquad \mu_n \xrightarrow[n\,\to\,\infty]{*} \mu$$

if and only if, for any $f \in \mathcal{C}$, we have that

$$\lim_{n\to\infty} \int_X f(x)\, d\mu_n = \int_X f(x)\, d\mu.$$

[If we could take $f = \chi_A$ (remark though that the characteristic function is *not* a continuous function!), since $\int \chi_A(x) d\mu = \mu(A)$, the definition above would imply that $\mu_n(A) \to \mu(A)$. If $A \in \mathcal{B}$ are *nice* sets (for example the boundary has measure zero) then the characteristic function χ_A can be approximated by continous functions and this is indeed the case.]

The two *facts* that we will assume without a proof are the following:

(F1) Two measures $\mu, \nu \in \mathcal{M}(X)$ are equal (i.e. $\mu = \nu$) if and only if

$$\int_X f(x)\, d\mu = \int_X f(x)\, d\nu \qquad \forall\, f \in \mathcal{C}(X).$$

(F2) The space $\mathcal{P}_T$ is (subsequentially) *compact*, i.e. for any sequence $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{P}_T$, there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ and a measure $\mu \in \mathcal{P}(X)$ such that $\lim_{k\to\infty} \mu_{n_k} = \mu$, i.e. the subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ converges weak star to μ.

[In the language of functional analysis, the first property, $(F1)$, says that the *algebra* of continuous functions $\mathcal{C}(X)$ *separates points*. This can be proved for example by using *Riesz representation theory*. The second property, $(F2)$, states that the unit ball is compact in the weak star topology and is a special case of a result known as *Banach-Alaouglu theorem*.]

Using these two *facts*, we can now show the existence of invariant measures for topological dynamical systems.

Proof of Krylov-Bogolyubov theorem, Thm 3.5.1. Pick any initial point $x \in X$, consider its (forward) orbit $\mathcal{O}_T^+(x)$ and define a sequence of measures by putting delta masses (Dirac measures) on initial segments of the orbit, namely set

$$\mu_n := \frac{1}{n}\sum_{i=0}^{n} \delta_{T^i(x)}, \qquad n \in \mathbb{N}.$$

Thus, for any given (Borel) measurable set $B \in \mathcal{B}$ we have that

$$\mu_n(B) = \frac{1}{n} Card\{0 \leq i < n, \text{ s.t. } \quad T^i(x) \in B\}.$$

[For this reason, μ_n is sometimes called a *counting measure* (it *counts* how many times the initial segment of lenght n of $\mathcal{O}_T^+(x)$ visits a given set).] In particular, one can see that for each $n \in \mathbb{N}$ μ_n is a *probability* measure, so $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{P}_T(X)$. It follows by the compactness in $(F2)$ that there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ and a measure $\mu \in \mathcal{P}(X)$ such that $\lim_{k\to\infty} \mu_{n_k} = \mu$ (where the convergence is in the weak star sense of Definition 3.5.1). Clearly μ is still a probability measure, i.e. $\mu \in \mathcal{P}(X)$.

We claim that the limit measure μ is actually *invariant* under T, i.e. $\mu \in \mathcal{P}_T(X)$. To show invariance, or equivalently that $T_*\mu = \mu$, by $(F1)$ it is enough to show that

$$\int_X f(x) dT^*\mu = \int_X f(x)\, d\mu, \qquad \forall f \in \mathcal{C}(X).$$

We claim that, by definition of $T^*\mu$, this means showing that

$$\int_X f(x) dT^*\mu = \int_X f \circ T(x)\, d\mu = \int_X f(x)\, d\mu, \qquad \forall f \in \mathcal{C}(X). \tag{3.13}$$

To see this, one can verify it for characteristic functions $f = \chi_A$, $A \in \mathscr{B}$ first (using that $T^*\mu = \mu(T^{-1}(A)$ and $\chi_A \circ T = \chi_{T^{-1}(A)}$), then extend it to simple functions and finally to any (positive) continuous function[7].

Using (in order) first (in (3.14)) the definition of $\mu = \lim_k \mu_{n_k}$ and of weak star convergence (see Definition 3.5.1), then recalling the definition of the measures μ_n as counting measures along pieces of the orbit (to get (3.15)), and finally using that $\int f\, d\delta_x = f(x)$ (to get (3.16)), we have that:

$$\begin{aligned}
\left| \int_X (f\circ T - f)(x)\, d\mu \right| &= \lim_{k\to\infty} \left| \int_X (f\circ T - f)(x)\, d\mu_{n_k} \right| &(3.14)\\
&= \lim_{k\to\infty} \left| \int_X (f\circ T - f)(x)\, d\left(\frac{1}{n_k}\sum_{i=0}^{n_k-1} \delta_{T^i(x)} \right) \right| &(3.15)\\
&= \lim_{k\to\infty} \frac{1}{n_k} \left| \sum_{i=0}^{n_k-1} (f\circ T - f)\left(T^i(x)\right) \right|. &(3.16)
\end{aligned}$$

Remark that in the (3.14) above we used that T is continuous, so that $f \circ T$ (and hence $f \circ T - f$) are again continuous functions and we can use them as *test functions* when we apply the definition of weak star convergence.

Now, the sum which appears in (3.16) is a *telescopic sum*, so that we can write:

$$\begin{aligned}
\sum_{i=0}^{n_k-1} (f\circ T - f)\left(T^i(x)\right) &= f(T(x)) - f(x) + f\left(T^2(x)\right) - f(T(x)) + \cdots + f\left(T^{n_k}(x)\right) - f\left(T^{n_k-1}(x)\right)\\
&= f(x) - f(T^{n_k-1}(x)).
\end{aligned}$$

[7]One can also notice that (3.13) is (yet another) variation of the Lemma on invariance via invariant functions: instead than considering all measurable functions, it is enough to restrict to continuous ones since in our setup $\mathcal{C}(X)$ is dense in the space of integrable function (with respect to the supremum norm $\|\|\cdot\|\|_\infty$.

Thus, since f is continous on a compact set X and hence *bounded* (n absolute value) by a constant $C_f > 0$ (we can take $C_f := \max_{x\in X} |f(x)|$), and as $k \to \infty$ also $n_k \to \infty$,

$$0 \leq \left| \int_X (f \circ T - f)(x)\, d\mu \right| = \lim_{n_k\to\infty} \frac{|f(x) - f(T^{n_k-1}(x))|}{n_k} \leq \lim_{n_k\to\infty} \frac{2C_f}{n_k} = 0.$$

This shows that (3.13) holds and hence that $\mu \in \mathcal{M}_T(X)$ concluding the proof that $\mathcal{M}_T(X) \neq \emptyset$. □

Remark that we have already seen an example of an *invariant measure* obtained form a (periodic) orbit: if $x \in Per_n(T)$, the counting measure $\frac{1}{n}\sum_{k=0}^{n-1} \delta_{T^i(x)}$ is invariant (this was proved as an exercise). In general, if x is not a periodic point (but on the other hand has a dense orbit), the measures μ_n converge towards a measure which is not discrete any more. We will see after proving the Birkhoff ergodic theorem that (when T is ergodic) for μ-a.e. initial point x, the limit measure μ (and more in general any weak limit of the sequence) is the invariant measure μ[8].

3.6 Ergodic Transformations

In this lecture we will define the notion of *ergodicity*, or *metric indecomposability*. Ergodic measure-preserving transformations are the building blocks of all measure-preserving transformations (as prime numbers are building blocks of natural numbers). Moreover, ergodicity will play an important role in Birkhoff ergodic theorem.

Let $(X, \mathscr{A}, \mu)$ be a *finite* measure space. In this section (and more in general when we want to talk of ergodic transformations) we will assume that $(X, \mathscr{A}, \mu)$ is a *probability space*. This is not a great restriction, since if $\mu(X) < \infty$, if we consider $\mu/\mu(X)$ (that is, the measure *rescaled* by $\mu(X)$), then $\mu/\mu(X)$ is a measure with total mass 1 and $(X, \mathscr{A}, \mu/\mu(X))$ is a probability space. Let $T : X \to X$ be a measure-preserving transformation.

Definition 3.6.1. A set $A \subset X$ is called *invariant* invariant under T (or simply invariant if the transformation is clear from the context) if

$$T^{-1}(A) = A.$$

Remark that in the definition we consider preimages T^{-1}. This is important if the transformation is not invertible.

Exercise 3.6.1. If T is invertible, show that A is invariant if and only if $T(A) = A$.

Example 3.6.1. Assume that T is invertible and that $x \in X$ is a periodic point of period n. Then

$$A = \{x, T(x), \ldots, T^{n-1}(x)\} \tag{3.17}$$

is an invariant set.

Definition 3.6.2. A measure preserving transformation T on a probability space $(X, \mathscr{A}, \mu)$ is *ergodic* if and only if for any set measurable $A \in \mathscr{A}$ such that $T^{-1}(A) = A$ either $\mu(A) = 0$ or $\mu(A) = 1$, that is *all invariant sets are trivial* from the point of view of the measure.

[8] We say in this case that x is *equdistributed with respect to* μ. If x is neither periodic, nor equidistributed with respect to μ (which is the case for μ-a.e. point when T is ergodic), μ_n might converge to *another* invariant measure, or not converge at all. The points for which the corresponding sequence $(\mu_n)_{n\in\mathbb{N}}$ converge to some $\nu \in \mathcal{M}_T(X)$ for the *basic of attraction* of the measure ν. There are classes of dynamical systems for which one is able to completely classify invariant measures and the set of points which converge to each of them. These are important classes of dynamical systems, since in this case one can predict the (asymptotic) behaviour of *each* orbit, while usually in ergodic theory one can only discuss full measure sets of initial conditions.

Remark 3.6.1. A transformation which is *not* ergodic is reducible in the following sense. If $A \in \mathscr{A}$ is an *invariant* measurable set of positive measure $\mu(A) > 0$, then we can consider the restriction μ_A of the measure μ to A, that is the measure defined by

$$\mu_A(B) = \frac{\mu(A \cap B)}{\mu(A)}, \qquad \text{for all } B \in \mathscr{A}.$$

It is easy to check that μ_A is again a probability measure and that it is invariant under T (Exercise). Remark that we used that $\mu(A) > 0$ to renormalize μ_A. Similarly, also the restriction $\mu_{X\backslash A}$ of the measure μ to the complement $X\backslash A$, given by

$$\mu_{X\backslash A}(B) = \frac{\mu(X\backslash A \cap B)}{\mu(X\backslash A)}, \qquad \text{for all } B \in \mathscr{A},$$

is an invariant probability measure (Exercise). Remark that here we used that $\mu(X\backslash A) > 0$ since $\mu(A) < 1$. Thus, we have decomposed μ into two invariant measures μ_A and $\mu_{X\backslash A}$ and one can study separately the two dynamical systems obtained restricting T to A and to $X\backslash A$. In this sense, non ergodic transformations are *decomposable* while ergodic transformations are *indecomposable*.

As prime numbers, that cannot be written as product of prime numbers, are the basic building block used to decompose any other integer number, similarly ergodic transformations, that are indecomposable in this metric sense, are the basic building block used to study any other measure-preserving transformation.

Exercise 3.6.2. Let $(X, \mathscr{A})$ be a measurable space and $T : X \to X$ be a transformation.

(a) Check that if μ_1 and μ_2 are probability measures on $(X, \mathscr{A})$, then any linear combination

$$\mu = \lambda\mu_1 + (1-\lambda)\mu_2, \qquad \text{where} \quad 0 \le \lambda \le 1,$$

is again a measure. Check that it is a probability measure.

(b) Let μ be a measure on $(X, \mathscr{A})$ preserved by T. Let $A \in \mathscr{A}$ be a measurable set with positive measure $\mu(A) > 0$. Check that by setting

$$\mu_1(B) = \frac{\mu(A \cap B)}{\mu(A)} \quad \text{for all } B \in \mathscr{A}, \qquad \mu_2(B) = \frac{\mu(A^c \cap B)}{\mu(A^c)} \quad \text{for all } B \in \mathscr{A},$$

(where $A^c = X\backslash A$ denotes the complement of A in X) one defines two probability measures μ_1 and μ_2. Show that if A is *invariant* under T, then both μ_1 and μ_2 are invariant under T.

(c) Show using the two previous points that a probability measure μ invariant under T is *ergodic* if it cannot be written as strict linear combination of two invariant probability measures for T, that is as

$$\mu = \lambda\mu_1 + (1-\lambda)\mu_2, \qquad \text{where} \quad 0 < \lambda < 1, \quad \mu_1 \neq \mu_2. \tag{3.18}$$

[The converse is also true, but harder to prove: a measure μ is ergodic *if and only* if it cannot be decomposed as in (3.18).]

Part (a) of Exercise 3.6.2 shows that the space of all probability T-invariant measures is *convex* (recall that a set C is a convex if for any $x, y \in C$ and any $0 \le \lambda \le 1$ the points $\lambda x + (1-\lambda)\lambda y$ all belong to C). If C is a convex set, the *extremal points* of C are the points $x \in C$ which cannot be expressed as linear combination of the other points, that is, there is no $0 < \lambda < 1$ and $x_1 \neq x_2$ such that $x = \lambda x_1 + (1-\lambda)x_2$. Thus, Part (b) of Exercise 3.6.2 shows that *ergodic probability measures are extremal points* of the set of all probability T-invariant measures.

Let us now give an example of a *non-ergodic* transformation and one of an ergodic one.

Example 3.6.2. [**Rational rotations are not ergodic**] Let $X = S^1$, $\mathscr{B}$ its Borel subsets and λ the Lebesgue measure. Consider a rational rotation $R_\alpha : S^1 \to S^1$ where $\alpha = p/q$ with p, q coprime. Consider for example the following set in $\mathbb{R}/\mathbb{Z}$

$$A = \bigcup_{i=0}^{q-1} \left[\frac{i}{q}, \frac{i}{q} + \frac{1}{2q}\right].$$

The set A in S^1 is shown in Figure 3.2.

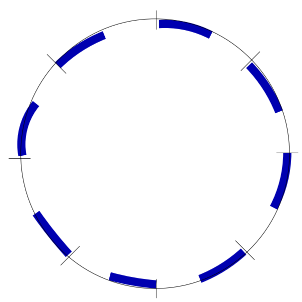


Figure 3.2: An invariant set A with $0 < \lambda < 1$ for a rational rotation $R_{p/q}$ with $q = 8$.

The set A is clearly invariant under $R_{p/q}$, since the clockwise rotation $R_{p/q}$ by $2\pi p/q$ sends each interval into another one. Since A is union of q intervals of equal length $1/2q$, $\lambda(A) = 1/2$, so $0 < \lambda(A) < 1$. Thus, since we constructed an invariant set whose measure is neither 0 nor 1, $R_{p/q}$ is *not* ergodic.

[Remark that any point is periodic of period q and since R_α is invertible, any periodic orbit is an invariant set, but it has measure zero. Thus, to show that $R_{p/q}$ is *not* ergodic, we need to construct an invariant set with *positive* measure and here we constructed one by considering the orbit of an interval.]

In the next lecture we will prove that on the other hand *irrational rotations* are ergodic. Thus, a rotation R_α is ergodic if and only if α is irrational.

Let us show that the doubling map is ergodic directly using the definition of ergodicity.

Example 3.6.3. [**The doubling map is ergodic**] Let $X = \mathbb{R}/\mathbb{Z}$, $\mathscr{B}$ be Borel sets of $\mathbb{R}$ and λ the Lebesgue measure. Let $T : X \to X$ be the doubling map, that is $T(x) = 2x \mod 1$. Let us show that the doubling map is ergodic.

Let $A \in \mathscr{B}$ be an invariant set, so that $T^{-1}(A) = A$. We have to show that $\lambda(A)$ is either 0 or 1. If $\lambda(A) = 1$, we are done. Let us assume that $\lambda(A) < 1$ and show that then $\lambda(A)$ has to be 0. Since we assume that $\lambda(A) < 1$, $\lambda(X\backslash A) > 0$. One can show (see Theorem 3.6.1 in the Extra) that a measurable set of positive measure is well approximated by small intervals in the following sense: given $\epsilon > 0$, we can find $n \in \mathbb{N}$ and a dyadic interval I of length $1/2^n$ such that

$$\lambda(I\backslash A) > (1-\epsilon)\lambda(I), \tag{3.19}$$

that is, the proportion of points in I which are *not* is A is at least $1-\epsilon$.Recall that we showed that if I is a dyadic interval of length $1/2^n$, its images $T^k(I)$ under the doubling map for $0 \le k \le n$ are again dyadic intervals of size $1/2^{n-k}$. In particular, the length $\lambda(T^k(I))$ is $2^k\lambda(I)$ and for $k = n$, $T^n(I) = \mathbb{R}/\mathbb{Z} = X$. Furthermore we can calculate that

$$\lambda(T^n(I\backslash A)) = 2^n\lambda(I\backslash A) = (1-\epsilon).$$

We now need to show that $T^n(I\backslash A) \subseteq X\backslash A$. To do this suppose $x \in T^n(I\backslash A)$ and $x \in A$. We then have that there exists $y \in I\backslash A$ such that $T^n(y) = x$ and therefore $y \in T^{-n}(A) = A$. This is a contradiction since

$y \in I \backslash A$. Therefore there is no such x and $T^n(I\backslash A) \subseteq X\backslash A$. Putting this together means that

$$\lambda(X\backslash A) \geq \lambda(T^n(I\backslash A)) \geq 1 - \epsilon.$$

Since $\lambda(X\backslash A) \geq 1 - \epsilon$ holds for all $\epsilon > 0$, we conclude that $\lambda(X\backslash A) = 1$ and hence $\lambda(A) = 0$. This concludes the proof that the doubling map is ergodic.

To prove directly from the definition that the doubling map is ergodic, we had to use a fact from measure theory that we stated without a proof (the existence of the interval in (3.19), see also Theorem 3.6.1 in the Extra). In the following lecture, §3.6, we will see how to prove ergodicity using Fourier series and we will see that is possible to reprove that the doubling map is ergodic by using Fourier series, which gives a simpler and self-contained proof.

Exercise 3.6.3. Let $X = [0, 1]$, $\mathscr{B}$ the Borel $\sigma-$algebra, λ the $1-$dimensional Lebesgue measure. Let $m > 1$ is an integer and consider the linear expanding map $T_m(x) = mx \mod 1$. Show that T_m is ergodic.
[*Hint*: mimic the previous proof that the doubling map is ergodic.]

Ergodicity via invariant functions

The following equivalent definition of ergodicity is also very useful to prove that a transformation is ergodic:

Lemma 3.6.1 (Ergodicity via measurable invariant functions). *A measure preserving transformation $T : X \to X$ is ergodic if and only if, any measurable function $f : X \to \mathbb{R}$ that is invariant, that is such that*

$$f \circ T = f \qquad \textit{almost everywhere} \quad (\textit{that is,} \;\; f(T(x)) = f(x) \quad \textit{for } \mu - \textit{almost every } x \in X) \tag{3.20}$$

is μ-almost everywhere constant (that is, there exists $c \in \mathbb{R}$ such that $f(x) = c$ for μ-a.e. $x \in X$).

Proof. Assume first that (3.20) hold. Let $A \in \mathscr{B}$ be an invariant set. Consider its characteristic function χ_A, which is measurable (see Example 3.4.1 in §3.4). Let us check that χ_A is an invariant *function*, that is $\chi_A \circ T = \chi_A$. Recall that we showed last time that $\chi_A \circ T = \chi_{T^{-1}(A)}$ (see equation (3.14) in §3.4). Thus

$$\chi_A \circ T = \chi_{T^{-1}(A)} = \chi_A, \qquad (\text{since } \; T^{-1}(A) = A).$$

Thus, we can apply (3.20) to χ_A and conclude that χ_A is almost everywhere constant. But since an indicatrix function takes only the values 0 and 1, either

$$\chi_A = 0 \; a.e. \qquad \Rightarrow \qquad \mu(A) = \int_A \chi_A \mathrm{d}\mu = 0, \qquad \text{or} \tag{3.21}$$

$$\chi_A = 1 \; a.e. \qquad \Rightarrow \qquad \mu(A) = \int_A \chi_A \mathrm{d}\mu = 1. \tag{3.22}$$

This concludes the proof that T is ergodic.

Let us assume now that T is ergodic and prove (3.20). Let $f : X \to \mathbb{R}$ be a measurable function. Assume that $f \circ T = f$ almost everywhere. One can redefine f on a set of measure zero so that the redefined function, which we will still call f, is invariant everywhere, that is $f(T(x) = f(x)$ for all $x \in X$ (see Exercise 3.6.4).

Consider the sets

$$A_t = \{x \in X, \text{ such that } f(x) > t\}, \qquad t \in \mathbb{R}.$$

The set A_t are called *level sets* of the function f and they are measurable since $A_t = f^{-1}((t, +\infty))$ and f is measurable, which by definition means that the preimage of each interval is in $\mathscr{B}$. Let us show that each A_t is *invariant*:

$$\begin{aligned}
T^{-1}(A_t) &= \{x \in X, \text{ such that } T(x) \in A_t\} && (\text{by definition of preimage}) \\
&= \{x \in X, \text{ such that } f(T(x)) > t\} && (\text{by definition of } T(x) \in A_t) \\
&= \{x \in X, \text{ such that } f(x) > t\} && (\text{since } f(T(x)) = f(x)) \\
&= A_t.
\end{aligned}$$

Thus, since T is ergodic, for each $t \in \mathbb{R}$ either $\mu(A_t) = 0$ or $\mu(A_t) = 1$. If the function f is constant equal to c almost everywhere, then $\mu(A_t) = 1$ for all $t < c$ (since $f(x) = c > t$ for a.e. $x \in X$), while $\mu(A_t) = 0$ for all $t \geq c$ (since $f(x) = c \leq t$ for a.e. $x \in X$). On the other hand, if f is *not* constant almost everywhere, one can find a level set t_0 such that $0 < \mu(t_0) < 1$, which is a contradiction with what we just proved. Thus, f has to be constant almost everywhere. This shows that (3.20) holds when T is ergodic. □

Exercise 3.6.4. Let $T : X \to X$ be a measure preserving transformation of the measured space $(X, \mathscr{B}, \mu)$. Let f be a measurable function $f : X \to \mathbb{R}$ that is invariant almost everywhere under T, that is $f \circ T(x) = f(x)$ for μ-almost every $x \in X$.

(a) Consider the set

$$E = \bigcup_{n \in \mathbb{N}} T^{-n}(N), \qquad \text{where} \quad N = \{ \ x \text{ such that } f(T(x)) \neq f(x) \ \}.$$

Show that $\mu(E) = 0$ and that $T^{-1}(E) \subset E$.

(b) Define a new function $\tilde{f}$ by setting:

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \notin E \\ 0 & \text{if } x \in E, \end{cases}$$

Show that $f = \tilde{f}$ μ-almost everywhere and that $\tilde{f} \circ T = \tilde{f}$ holds everywhere, that is $\tilde{f}(T(x)) = \tilde{f}(x)$ for *all* $x \in X$.

Exercise 3.6.5. Let $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a rational rotation, where $\alpha = p/q$ and p, q are coprime. Show that R_α is *not* ergodic by exhibiting a non-constant invariant function.

One can show that in Lemma 3.6.1 instead then considering all functions which are measurable, it is enough to check that (3.20) holds for all integrable functions $f \in L^1(\mu)$ or for all square-integrable functions $f \in L^2(\mu)$ (the definition of these spaces was given in §3.4 We have the following two variants of Lemma 3.6.1:

Lemma 3.6.2 (Ergodicity via invariant integrable functions). *Let $(X, \mathscr{B}, \mu)$ be a probability space and $T : X \to X$ a measure preserving transformation. Then $T : X \to X$ is ergodic if and only if*

$$\text{for all } f \in L^1(X, \mathscr{B}, \mu), \quad f \circ T = f \ \mu - a.e. \quad \Rightarrow \quad f \ \mu - a.e. \text{ constant.}$$

Lemma 3.6.3 (Ergodicity via invariant square integrable functions). *Let $(X, \mathscr{B}, \mu)$ be a probability space and $T : X \to X$ a measure preserving transformation. Then $T : X \to X$ is ergodic if and only if*

$$\text{for all } f \in L^2(X, \mathscr{B}, \mu), \quad f \circ T = f \ \mu - a.e. \quad \Rightarrow \quad f \ \mu - a.e. \text{ constant.}$$

Exercise 3.6.6. Let $(X, \mathscr{B}, \mu)$ be a measured space and $T : X \to X$ be a *measure-preserving transformation.* Consider the space $L^2(X, \mathscr{B}, \mu)$ of square integrable functions.

Let $U_T : L^2(X, \mathscr{B}, \mu) \to L^2(X, \mathscr{B}, \mu)$ be given by

$$U_T(f) = f \circ T.$$

(a) Check that $L^2(X, \mathscr{B}, \mu)$ is a vector space and U_T is *linear*, that is for all $f, g \in L^2(\mu)$ and $a, b \in \mathbb{R}$,

$$af + bg \in L^2(X, \mathscr{B}, \mu) \quad \text{and} \quad U_T(af + bg) = aU_T(f) + bU_T(g).$$

(b) Show that U_T preserves the $L^2(\mu)-$norm, that is for any $f \in L^2(\mu)$ we have $\|U_T(f)\|_2 = \|f\|_2$. Deduce that if $d : L^2(\mu) \times L^2(\mu) \to \mathbb{R}^+$ is the distance given by

$$d(f, g) = \|f - g\|_2, \quad \text{for all} \quad f, g \in L^2(\mu),$$

U_T is an *isometry*, that is

$$d(U_T(f), U_T(g)) = d(f, g) \qquad \text{for all} \quad f, g \in L^2(\mu);$$

(c) Verify that if a function is constant almost everywhere, then it is an eigenvector of U_T with eigenvalue 1. Assume in addition that $(X, \mathscr{B}, \mu)$ is a *probability space*. Show that T is ergodic if and only if the *only* eigenfunctions $f \in L^2(\mu)$ of U_T corresponding to the eigenvalue 1 are constant functions.

[*Hint:* Both Part (b) and (c) of the exercise consist only of recalling and re-interpreting definitions.]

The operator U_T is known as the *Koopman operator* associated to T. Many ergodic properties can be equivalently rephrased in terms of properties of the operator U_T, as ergodicity in Part (c). The study of the properties of U_T and its spectrum (for example, its eigenvalues) is known as *spectral theory of dynamical systems.*

Extra: Lebesgue density points

In the proof that the doubling map is ergodic, we used the following Theorem from measure theory, known as Lebesgue density Theorem.

Let $X = \mathbb{R}^n$ and λ be $n-$dimensional Lebesgue measure. Let $A \subset \mathbb{R}^n$ be a Borel measurable set. Let $B(x, \epsilon)$ denote the ball of radius ϵ at x. The *density* of A at x, denoted by $d_x(A)$, is by definition

$$d_x(A) = \lim_{\epsilon \to 0} \frac{\lambda(A \cap B(x, \epsilon))}{\lambda(B(x, \epsilon))}.$$

A point $x \in A$ is called a *Lebesgue density point* for A if the density $d_x(A) = 1$. Thus, if x is a density point, small intervals containing x intersect A on a large proportion of their measure, tending to 1 as the size of the interval tends to zero.

Theorem 3.6.1 (Lebesgue density). *Let $X = \mathbb{R}^n$ and λ be $n-$dimensional Lebesgue measure. If $A \subset \mathbb{R}^n$ is a Borel measurable set with positive measure $\lambda(A) > 0$, almost every point $x \in A$ is a Lebesgue density point for A.*

This theorem implies that measurable sets can be well approximated by small intervals: on a small scale, measurable sets fill densely the space, so if I is sufficiently small and intersects the set A, most of the points in I are contained in A (only a set of points whose measure is a proportion ϵ of the total measure is left out). Similarly, other small intervals will be missed almost completely by the set A, so that the set can be approximated well by a union of small intervals.

Exercise 3.6.7. Deduce from the Lebesgue density Theorem the fact that we used in the proof that the doubling map is ergodic, that is: if $\mu(X \backslash A) > 0$, given any $\epsilon > 0$, we can find $n \in \mathbb{N}$ and a dyadic interval I of length $1/2^n$ such that

$$\lambda(I \backslash A) > (1 - \epsilon)\lambda(I).$$

3.7 Ergodicity using Fourier Series

In the previous lecture §3.5 we defined *ergodicity* and showed from the definition that the doubling map is ergodic. In this lecture we will show how to use Fourier Series to show that certain measure preserving transformations defined on $\mathbb{R}/\mathbb{Z}$ or on the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ are ergodic. Proving ergodicity using Fourier Series turns out to be very simple and elegant. We first give a brief overview of the basics of Fourier Series.

Let $(X, \mathscr{B}, \mu)$ be a measure space. In §3.4 we defined integrals with respect to a measure. Recall that we also introduced the following notation for the spaces of *integrable* and *square-integrable functions*

$$L^1(X, \mathscr{B}, \mu) = L^1(\mu) = \{f : X \to \mathbb{R},\ f \text{ measurable}, \int |f| \mathrm{d}\mu < +\infty\}/\sim,$$

$$L^2(X, \mathscr{B}, \mu) = L^2(\mu) = \{f : X \to \mathbb{R},\ f \text{ measurable}, \int |f|^2 \mathrm{d}\mu < +\infty\}/\sim,$$

where $f \sim g$ if $f = g$ μ-almost everywhere and the norms are respectively given by $\|f\|_1 := \int |f| \mathrm{d}\mu$ and $\|f\|_2 := (\int |f|^2 \mathrm{d}\mu)^{1/2}$.

Fourier Series

Warning: we just started the very beginning of this section only in class, it will be continued next week. The whole section is included for your convenience, if you want to read it in advance.

Let $X = \mathbb{R}/\mathbb{Z}$ with the Lebesgue measure λ. Consider a function $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ (more in general, one can consider a function $f : \mathbb{R} \to \mathbb{R}$ which is 1−periodic, that is $f(x + 1) = f(x)$ for all $x \in \mathbb{R}$). We would like to represent f as superposition of harmonics, decomposing it via the basic oscillating functions

$$\sin(2\pi nx), \quad \cos(2\pi nx), \qquad n = 0, 1, 2, \ldots,$$

More precisely, we would like to represent f as a linear combination

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi nx) + \sum_{n=1}^{\infty} b_n \sin(2\pi nx).$$

Instead than using this notation, we prefer to use the complex notation, which is more compact. Recall the identity

$$e^{2\pi ix} = \cos(2\pi nx) + i \sin(2\pi nx).$$

Using this identity, one can can show that we can equivalently write

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi nx) + \sum_{n=1}^{\infty} b_n \sin(2\pi nx) = \sum_{n=-\infty}^{+\infty} c_n e^{2\pi inx}, \tag{3.23}$$

$$\text{where } c_n = \begin{cases} \frac{1}{2}(a_n - ib_n) & \text{if } n > 0, \\ a_0/2 & \text{if } n = 0, \\ \frac{1}{2}(a_{-n} + ib_{-n}) & \text{if } n < 0. \end{cases} \tag{3.24}$$

Thus, we look for a representation of f of the form

$$\sum_{n=-\infty}^{+\infty} c_n e^{2\pi inx}.$$

[Using this complex form, more in general, one can try to represent more in general functions $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$. See also Exercise 3.7.1, Part (b)].

Exercise 3.7.1. (a) Verify that if a_n, b_n and c_n are related by (3.24), then (3.23) holds.
Assume that

$$f = \sum_{n=-\infty}^{+\infty} c_n e^{2\pi inx}.$$

(b) Show that f is *real* if and only if $c_{-n} = \overline{c_n}$ for all $n \in \mathbb{Z}$ (where $\overline{z}$ denotes the complex conjugate of z);

(c) Show that f is *even* (that is $f(-x) = f(x)$ for all x) if and only if $c_n = c_{-n}$ for all $n \in \mathbb{Z}$; show that f is *odd* (that is $f(-x) = -f(x)$ for all x) if and only if $c_n = -c_{-n}$ for all $n \in \mathbb{Z}$.

Definition 3.7.1. If $f \in L^1(\mathbb{R}/\mathbb{Z}, \mathscr{B}, \mu)$ we say that the *Fourier series* of f is the expression

$$\sum_{n=-\infty}^{+\infty} c_n e^{2\pi inx}, \qquad \text{where} \quad c_n = \int f(x) e^{-2\pi inx} \mathrm{d}\mu, \; n \in \mathbb{Z}.$$

The c_n, $n \in \mathbb{Z}$, are called *Fourier coefficients* of f. Remark that $c_0 = \int f \mathrm{d}\mu$.

We denote by $S_N f$ the N^{th} partial sum of the Fourier series of f, given by

$$S_N f(x) = \sum_{n=-N}^{+N} c_n e^{2\pi inx}.$$

One needs the assumption $f \in L^1(\mu)$ to guarantee that the Fourier coefficients, and hence the Fourier series, is well-defined. The following property of the Fourier coefficients can be easily proved and is very helpful to use to prove ergodicity in certain examples (see Exercise 3.7.3):

Lemma 3.7.1 (Riemann-Lebesgue Lemma). *If $f \in L^1(\mu)$, the Fourier coefficients c_n in Definition 3.7.1 tend to zero in modulus, that is $|c_n| \to 0$ as $|n| \to \infty$.*

Unfortunately, as you should know well from the study of series, the fact that the coefficients tend to zero is not enough to guarantee that the Fourier series converges. It is natural to ask *when* the Fourier series *converges* for all points and when does it actually *represents* the function f from which we started and in which sense. We list below some answers to these questions.

Once we have a representation of f as a Fourier series (in one of the senses here below), one can use Fourier series as a tool which turns out to be very useful in applications. We will use them to show ergodicity, but more in general Fourier series can be used to solve differential equations and have a huge number of applications in applied mathematics.

The following can be proved:

(F1) If $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is *differentiable* and the derivative is *continuous*, than its Fourier series converges at every point:

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{2\pi i n x}, \qquad \text{for all } \ x \in \mathbb{R}/\mathbb{Z}.$$

We say in this case that $S_N f$ *converges pointwise* to f.

[Remark that if f is *only continuous* (but not necessarily differentiable), it is *not* necessarily *true* that the Fourier series of f converges to f pointwise. The proof of pointwise convergence can be found in many books in Real Analysis or Harmonic Analysis.]

(F2) If $f \in L^2(\mu)$,

$$\|S_N f - f\|_2 \to 0 \text{ as } N \to +\infty, \tag{3.25}$$

so that $S_N f$ approximate f better and better in the L^2-norm. We say in this case that the Fourier series *converges to f in L^2*.

[The proof of this statement is not hard and relies entirely on linear algebra. One can prove that $L^2(\mu)$ is a vector space and that the exponentials $e^{2\pi i n}$ form an orthogonal linear bases.]

(F3) If $f \in L^2(\lambda)$, one can actually show a much stronger statement (*Carlson's Theorem*):

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{2\pi i n x} \qquad \text{for a.e.} x \in \mathbb{R}/\mathbb{Z}.$$

[This result if very hard to prove, it was a hard open question and object of research for decades. The proof given by Carleson is very hard and many people have tried to understand it and simplify it.]

We will only use Fourier series for L^2-functions. A crucial property of Fourier series that we will use is *uniqueness*:

(F4) If $f \in L^1(\mu)$ and $c_n = 0$ for all $n \in \mathbb{Z}$, then $f = 0$. Recall that if μ is finite, $L^2(\mu) \subset L^1(\mu)$. As a consequence, if μ is a probability measure and $f \in L^2(\mu)$

$$\sum c_n e^{2\pi i n x} = \sum c'_n e^{2\pi i n x}$$

where the equality holds in the L^2 sense explained in $(F2)$, then $c_n = c'_n$ for all $n \in \mathbb{Z}$. Thus, the coefficients of the Fourier series of a function $f \in L^2(\mu)$ are *unique*.

Higher dimensional Fourier Series

Let $X = \mathbb{R}^d/\mathbb{Z}^d$. One can analogously want to represent a function $f : \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}$ as a linear combination of products of harmonics in all directions. A Fourier series in higher dimension (in complex notation) is an expression of the form

$$\sum_{n_1=-\infty}^{+\infty} \sum_{n_2=-\infty}^{+\infty} \cdots \sum_{n_d=-\infty}^{+\infty} c_{n_1,n_2,\dots,n_d} e^{2\pi i n_1 x_1} e^{2\pi i n_2 x_2} \dots e^{2\pi i n_d x_d}.$$

A more compact notation for this expression can be obtained by using vectors and scalar products: if we write

$$\underline{x} = (x_1, x_2, \dots, x_d), \quad \underline{n} = (n_1, n_2, \dots, n_d), \qquad < \underline{n}, \underline{x} >= n_1 x_1 + n_2 x_2 + \cdots + n_d x_d,$$

and write $c_{\underline{n}}$ for $c_{n_1,\dots,n_d}$, since $e^{2\pi i n_1 x_1} e^{2\pi i n_2 x_2} \dots e^{2\pi i n_d x_d} = e^{2\pi i (n_1 x_1 + \cdots + n_d x_d)}$, we can rewrite the above expression in the compact form

$$\sum_{\underline{n} \in \mathbb{Z}^d} c_{\underline{n}} e^{2\pi i < \underline{n}, \underline{x} >}.$$

If $f \in L^1(\mathbb{R}^d/\mathbb{Z}^d, \mathscr{B}, \mu)$ then we say that the above expression is the Fourier series of f if the coefficients $c_{\underline{n}}$ are given by

$$c_{\underline{n}} = \int f e^{-2\pi i < \underline{n}, \underline{x} >} \mathrm{d}\mu.$$

One can prove for these higher dimensional Fourier series results analogous to the 1 dimensional case. In particular:

(HF1) A result analogous to the *Riemann Lebesgue Lemma* holds: if $f \in L^1(\mu)$, the Fourier coefficients of f are such that $|c_{\underline{n}}| \to 0$ as $\underline{n} \to \infty$ (here $\underline{n} \to \infty$ means that the vector $\underline{n}$ tends to infinity in $\mathbb{R}^d$, for example its norm $\sqrt{n_1^2 + \cdots + n_d^2}$ grows).

(HF2) If $f \in L^2(\mu)$, the Fourier series of f converges to f in $L^2(\mu)$, in other words if we consider

$$S_N f = \sum_{n_1=-N}^{+N} \sum_{n_2=-N}^{+N} \cdots \sum_{n_d=-N}^{+N} c_{\underline{n}} e^{2\pi i < \underline{n}, \underline{x} >}$$

then $\|S_N f - f\|_2 \to 0$ as $N \to +\infty$;

(HF3) The Fourier coefficients of $f \in L^2(\mu)$ are unique, so if

$$\sum_{\underline{n} \in \mathbb{Z}^d} c_{\underline{n}} e^{2\pi i < \underline{n}, \underline{x} >} = \sum_{\underline{n} \in \mathbb{Z}^d} c'_{\underline{n}} e^{2\pi i < \underline{n}, \underline{x} >}$$

(where the equality holds in $L^2(\mu)$) then $c_{\underline{n}} = c'_{\underline{n}}$ for all vectors $\underline{n} \in \mathbb{Z}^d$.

Ergodicity of irrational rotations.

Consider the probability space $(\mathbb{R}/\mathbb{Z}, \mathscr{B}, \lambda)$. Let $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a rotation. We showed last time that if $\alpha = p/q$ is rational, $R_{p/q}$ is *not* ergodic. Using Fourier series, let us now show that if $\alpha \notin \mathbb{Q}$, then R_α *is* ergodic with respect to Lebesgue. Thus, rotations R_α are ergodic if and only if α is irrational.

Proof. Let us prove ergodicity by using L^2- invariant functions: given $f \in L^2(\lambda)$, we want to show that if $f \circ R_\alpha = f$ almost everywhere, then it is constant almost everywhere. By Lemma 3.6.3, this is equivalent to prove ergodicity. Since $f \in L^2(\mathbb{R}/Z, \mathscr{B}, \lambda)$, f is equal in L^2 to its Fourier series

$$\sum_{n=-\infty}^{+\infty} c_n e^{2\pi i n x}, \qquad \text{where} \quad c_n = \int f(x) e^{-2\pi i n x} \mathrm{d}x, \ n \in \mathbb{Z}.$$

By plugging $R_\alpha(x) = x + \alpha \mod 1$ in the Fourier series (and remarking that modulo 1 has no influence since $e^{2\pi ik} = 1$ for any $k \in \mathbb{Z}$) we get that the Fourier series for $F \circ R_\alpha$ is

$$\sum_{n=-\infty}^{+\infty} c_n e^{2\pi in(x+\alpha)} = \sum_{n=-\infty}^{+\infty} c_n e^{2\pi in\alpha} e^{2\pi inx}.$$

Since $f = f \circ R_\alpha$ almost everywhere we know that $f = f \circ R_\alpha$ in L^2 and so both Fourier series must be the same. Thus, by uniqueness of the Fourier coefficients, by equating the two different expressions for the n^{th} Fourier coefficient in front of $e^{2\pi inx}$, we must have

$$c_n = c_n e^{2\pi in\alpha} \quad \Leftrightarrow \quad c_n(1 - e^{2\pi in\alpha}) = 0 \qquad \text{for all } n \in \mathbb{Z}.$$

If $(1 - e^{2\pi in\alpha}) \neq 1$ we can then conclude that $c_n = 0$. Since α is irrational, all orbits of R_α are infinite and distinct, thus in particular the orbit $R_\alpha^n(0) = n\alpha \mod 1$ consists of distinct points. Thus, $n\alpha \mod 1 \neq 0$ for all $n \neq 0$, which shows that

$$\text{for all } n \neq 0, \quad (1 - e^{2\pi in\alpha}) \neq 1 \quad \Rightarrow \quad c_n = 0.$$

Thus all Fourier coefficients are zero apart from possibly c_0 and $f = c_0$ is constant. This concludes the proof of ergodicity. ☐

Exercise 3.7.2. (a) Where does the above proof fail if α is rational?

(b) Show that $R_{p/q}$ is not ergodic by exhibiting a function which is *invariant* but not constant;

(c) If $\alpha = p/q$ where p, q are coprime, what are all the possible Fourier series of invariant functions $f \in L^2(\mathbb{R}/\mathbb{Z}, \mathscr{B}, \lambda)$?

Ergodicity of linear expanding maps. Other measure-preserving transformations of $\mathbb{R}/\mathbb{Z}$ can be proved to be ergodic by using Fourier series. For example:

Exercise 3.7.3. Prove using Fourier series that the doubling map $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is ergodic with respect to λ Lebesgue.

[Hint: Use Riemann-Lebesgue Lemma.]

Exercise 3.7.4. Prove using Fourier series that any linear expanding map $T_m(x) = mx \mod 1$ on $\mathbb{R}/\mathbb{Z}$where $m > 1$ is an integer is ergodic with respect to the $1-$dimensional Lebesgue measure λ.

Ergodicity of toral automorphisms.

Let A be a $d \times d$ matrix with *integer entries* and determinant ± 1. Then A determines a toral automorphism $T_A : \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ of the $d-$dimensional torus by fist acting linearly by A and then taking the result modulo 1, as follows:

$$T_A(x_1, \dots, x_d) = \left(\sum_{j=1}^d A_{1j} x_j \mod 1, \sum_{j=1}^d A_{2j} x_j \mod 1, \dots, \sum_{j=1}^d A_{dj} x_j \mod 1 \right).$$

(The CAT map is an example where $d = 2$.) The proof that the map T_A is well defined is exactly the same than in the case $d = 2$. Consider as a measure space $(\mathbb{T}^d, \mathscr{B}, \lambda)$ where $\mathscr{B}$ are Borel sets and λ is the d-dimensional Lebesgue measure on the torus. Remark that $\lambda(\mathbb{T}^d) = 1$, so the space is a probability space. One can check that since $|det(A)| = 1$ the map T_A preserves the Lebesgue probability measure.

Using higher dimensional Fourier series, we will prove the following Theorem. Let us call a complex number $\lambda \in \mathbb{C}$ a *root of unity* if $\lambda^n = 1$ for some $n \in \mathbb{N}$. Remark that by solving the equation $\lambda^n = 1$ the n^{th} roots of unity are exactly the n complex numbers of modulus 1 and of the form

$$e^{2\pi ik/n}, \qquad k = 0, 1, \dots, n-1.$$

Theorem 3.7.1. *Let $T_A : \mathbb{T}^d \to \mathbb{T}^d$ be the toral automorphism determined by the matrix A. Then T_A is* ergodic *if and only if the matrix A has* no eigenvalues *which are* roots of unity.

As in $d = 2$, we call a toral automorphism T_A *hyperbolic* if the matrix A has no eigenvalues λ with $|\lambda| = 1$.

Corollary 3.7.1. *If $T_A : \mathbb{T}^d \to \mathbb{T}^d$ is a* hyperbolic *toral automorphism, then it is ergodic.*

Proof. The Corollary follows simply because if $\lambda^n = 1$, clearly $|\lambda| = 1$, so ruling out eigenvalues with modulus 1 automatically rules out in particular eigenvalues that are roots of unity. □

Example 3.7.1. The CAT map $T_A(x, y) = (2x + y \mod 1, x + y \mod 1)$ is *ergodic*, since it has eigenvalues $(1 \pm \sqrt{5})/2$.

Proof of Theorem 3.7.1. Assume that A has no roots of unity as eigenvalues. Let us prove ergodicity of T_A by using Fourier series. Let $f \in L^2(\mathbb{T}^d, \mathscr{B}, \lambda)$ be an invariant function, that is $f \circ T_A = f$ λ-almost everywhere. We want to show that f is constant λ-almost everywhere. Since $f \in L^2(\lambda)$, f is equal in $L^2(\lambda)$ to its Fourier series

$$\sum_{\underline{n}\in\mathbb{Z}^d} c_{\underline{n}} e^{2\pi i <\underline{n},\underline{x}>}, \qquad \text{where} \quad c_{\underline{n}} = \int f(x) e^{-2\pi i <\underline{n},\underline{x}>} \mathrm{d}\mu, \quad \underline{n} \in \mathbb{Z}^d. \tag{3.26}$$

Thus, we also have

$$f(T_A(\underline{x})) = \sum_{\underline{n}\in\mathbb{Z}^d} c_{\underline{n}} e^{2\pi i <\underline{n},A\underline{x}>} \tag{3.27}$$

(where we used that the modulo one part of the definition does not contribute to the coefficients, since if $\underline{k} \in \mathbb{Z}^d$, then $< \underline{n}, \underline{k} > \in \mathbb{Z}$ so $e^{2\pi i <\underline{n},\underline{k}>} = 1$). Remark that, if A^t denotes the transpose of the matrix A, we have

$$< \underline{n}, A\underline{x} > = < A^t\underline{n}, \underline{x} >, \tag{3.28}$$

since

$$\begin{aligned} < \underline{n}, A\underline{x} > &= \sum_{i=1}^{d} n_i, (A\underline{x})_i = \sum_{i=1}^{d} n_i \left(\sum_{j=1}^{d} A_{ij} x_j \right) = \sum_{i=1}^{d} \sum_{j=1}^{d} n_i A_{ij} x_j \\ &= \sum_{j=1}^{d} \sum_{i=1}^{d} A^t_{ji} n_i x_j = \sum_{j=1}^{d} \left(\sum_{i=1}^{d} A^t_{ji} n_i \right) x_j = \sum_{j=1}^{d} (A^t \underline{x})_j x_j = < A^t\underline{n}, \underline{x} > . \end{aligned}$$

Thus, using (3.28) we have the Fourier series for $f \circ T_A$

$$\sum_{\underline{n}\in\mathbb{Z}^d} c_{\underline{n}} e^{2\pi i <A^t\underline{n},\underline{x}>}. \tag{3.29}$$

Since $f = f \circ T_A$ by assumptions, the two Fourier series for f and $f \circ T_A$ are equal and since the Fourier coefficients are unique, the coefficients of the same terms must be the same. Consider the term $e^{2\pi i <A^t\underline{n},\underline{x}>}$ in (3.29), which has Fourier coefficient $c_{\underline{n}}$. The same term in (3.26) has Fourier coefficient $c_{A^t\underline{n}}$. Thus, we must have

$$c_{\underline{n}} = c_{A^t\underline{n}} \qquad \text{for all } \underline{n} \in \mathbb{Z}^d.$$

If for some $\underline{n} \in \mathbb{Z}^d$ we have $|c_{\underline{n}}| \neq 0$, then, since

$$c_{\underline{n}} = c_{A^t\underline{n}} = c_{(A^t)^2\underline{n}} = \cdots = c_{(A^t)^k\underline{n}} = \dots$$

all the terms $c_{(A^t)^k\underline{n}}$ for $k \in \mathbb{N}$ are equal and with modulus different than zero. If the indexes

$$\underline{n} = A^t\underline{n} = (A^t)^2\underline{n} = \cdots = (A^t)^k\underline{n} = \dots \tag{3.30}$$

are all distinct and thus there is an infinite number of coefficient with equal and non-zero modulus, this gives a contradiction, since otherwise the Fourier series would not converge (recall that by Riemann Lebesgue Lemma $|c_{\underline{m}}| \to 0$ as $\underline{m} \to \infty$). Thus, there can be only *finitely many* different terms in (3.30), which means that there exists $k_1 \neq k_2$ such that

$$(A^t)^{k_1}\underline{n} = (A^t)^{k_2}\underline{n} \qquad \Rightarrow \qquad (A^t)^k\underline{n} = \underline{n}, \quad \text{where } k = |k_1 - k_2|.$$

If $\underline{n}$ is not the zero vector $\underline{0} = (0, \ldots, 0)$, this shows that $\underline{n}$ is an eigenvector for $(A^t)^k$ with eigenvalue 1. Thus A^t must have an eigenvalue λ which is a root of unity (see the Remark below and apply it to the matrix A^t). But since the eigenvaleus of A^t are the same than the eigenvalues of A (since $det(A^t - \lambda Id) = det(A - \lambda Id)$), this would imply that also A has an eigenvalue which is a root of unity, which is excluded by assumption. Thus $\underline{n} = \underline{0}$.

conti

We showed that if $|c_{\underline{n}}| \neq 0$ then $\underline{n} = \underline{0}$. Thus, for all $\underline{n} \neq \underline{0}$, $c_{\underline{n}} = 0$ and $f = c_{\underline{0}}$. Thus, f is constant almost everywhere. This concludes the proof that T_A is ergodic if A has no eigenvalues which are roots of unity.

The converse implication is left as an exercise (Exercise 3.7.5). □

Remark 3.7.1. *To see that if A^k has eigenvector $\underline{v}$ with eigenvalue 1 then A has an eigenvalue which is a kth root of unity (when working in the complex numbers) take the subspace*

$$W = span\{\underline{v}, A\underline{v}, \ldots, A^{k-1}\underline{v}\}.$$

Note that for any $\underline{x} \in W$ we must have $A(\underline{x}) \in W$ so A must have an eigenvector $\underline{y} \in W$ with eigenvalue λ. We then know that $A^k\underline{y} = \lambda^k\underline{y}$. However for any $\underline{x} \in W$ we will have that $A^k\underline{x} = \underline{x}$ and thus $\lambda^k = 1$ and so λ is a kth root of union.

Exercise 3.7.5. Assume that the $d \times d$ integral matrix A has an eigenvalue λ which is a root of unity, that is, $\lambda^k = 1$ for some $k \in \mathbb{N}$.

(a) Show that there exists $\underline{n} \in \mathbb{Z}^d$ such that $\underline{n} \neq \underline{0}$ and

$$(A^t)^k\underline{n} = \underline{n},$$

where A^t denotes the transpose matrix of A;

[*Hint:* Show first that there exists $\underline{n} \in \mathbb{R}^d$ such that $\underline{n} \neq \underline{0}$ and $(A^t)^k\underline{n} = \underline{n}$. Use that A has integer entries to conclude that we can choose $\underline{n}$ in $\mathbb{Z}^d$.]

(b) Prove that the toral endomorphism $T_A : \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}^d/\mathbb{Z}^d$ associated to such a matrix A is *not* ergodic.

[*Hint:* Use Fourier series to construct a non-constant invariant function.]

Thus, this exercise proves the converse implication in Theorem 3.7.1: if A has an eigenvalue λ which is a root of unity, then T_A is *not* ergodic.

Other measure-preserving transformations of $\mathbb{R}^d/\mathbb{Z}^d$ can be proved to be ergodic by using higher-dimensional Fourier series.

Ergodicity of translations on the torus. Let $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{R}^d$. The translation by $\underline{\alpha}$ on the $d-$dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$is the transformation $R_{\underline{\alpha}} : \mathbb{T}^d \to \mathbb{T}^d$ given by

$$R_{\underline{\alpha}}(x_1, x_2, \ldots, x_d) = (x_1 + \alpha_1 \mod 1, x_2 + \alpha_2 \mod 1, \ldots, x_d + \alpha_d \mod 1).$$

Exercise 3.7.6. Show that $R_{\underline{\alpha}} : \mathbb{T}^d \to \mathbb{T}^d$ preserves the $d-$dimensional Lebesgue measure λ on Borel sets $\mathscr{B}$ on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.

[*Hint:* You can either verify the area-preserving relation first for rectangles or use the characterization of measure-preserving using functions and a change of variables.]

We say that the translation vector $\underline{\alpha}$ is *irrational* if the components $\alpha_1, \alpha_2, \ldots, \alpha_d$ are *rationally independent*, that is if

$$n_1\alpha_1 + n_2\alpha_2 + \cdots + n_d\alpha_d = k \text{ for some } n_1, n_2, \ldots, n_d, k \in \mathbb{Z} \quad \Rightarrow \quad n_1 = \cdots = n_d = k = 0.$$

Equivalently, $\underline{\alpha}$ is irrational is there is no $\underline{n} \in \mathbb{Z}^d \backslash \{\underline{0}\}$ such that $< \underline{n}, \underline{x} >= k$ for some integer $k \in \mathbb{Z}$.

Proposition 1. *The translation $R_{\underline{\alpha}}$ on $\mathbb{T}^d$ is ergodic if and only if $\underline{\alpha}$ is irrational.*

Exercise 3.7.7. Prove Proposition 1:

(a) using Fourier series, show that if there is no $\underline{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$, $\underline{n} \neq (0, \ldots, 0)$, such that

$$< \underline{n}, \underline{\alpha} >= n_1\alpha_1 + n_2\alpha_2 + \cdots + n_d\alpha_d = k \qquad \text{for some } k \in \mathbb{Z},$$

then $R_{\underline{\alpha}}$ is ergodic with respect to the $d-$dimentional Lebesgue measure λ;

(b) If there exists $\underline{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$ such that $\underline{n} \neq \underline{0}$ and $< \underline{n}, \underline{\alpha} > \in \mathbb{Z}$, show that $R_{\underline{\alpha}}$ is *not ergodic.*

[*Hint*: Look for a non-constant invariant function $f : X \to \mathbb{C}$ whose Fourier series has only one term. You can then use it to find a real-valued non-constant invariant function.]

Ergodicity of the skew product over an irrational rotation. Let $X = \mathbb{T}^2$ (with the Borel $\sigma-$algebra and the Lebesgue measure λ. Let $\alpha \in \mathbb{R}$ and consider the map $T : \mathbb{T}^2 \to \mathbb{T}^2$ given by

$$T(x, y) = (x + \alpha \mod 1, x + y \mod 1). \tag{3.31}$$

Exercise 3.7.8. Check that T preserves the two dimensional Lebesgue measure λ. [*Hint:* You can use Fubini theorem.]

A map T of the form (3.31) above, where the first coordinate of $T(x, y)$ is a function only of the first coordinate x, is called a *skew shift*. Since the T acts as a rotation on the first coordinate, this map is called a *skew shift over a rotation.*

Proposition 2. *The skew shift T over the rotation R_α is* ergodic *with respect to* λ if and only if *the R_α is ergodic. Equivalently, T is ergodic if and only if α is irrational.*

Exercise 3.7.9. Prove Proposition 2:

(a) Use Fourier series to show that if α is irrational then T is ergodic;

(b) Show that if α is rational then T there is a non-constant invariant function, thus T is not ergodic.

3.8 Birkhoff (pointwise) Ergodic Theorem

In this section we will state the Birkhoff Ergodic Theorem, which is one of the key theorems in Ergodic Theory. The motivation came originally from the *Boltzmann Ergodic Hypothesis* formulated by Boltzmann in the 1930s (see below). The concept of ergodicity was developed exactly in order to prove the Boltzmann Ergodic Hypothesis, thus giving birth to the field of Ergodic Theory.

Botzmann Ergodic Hypothesis

Let X be the phase space of a physical system (for example, the points of X could represent configurations of positions and velocities of particles of a gas in a box). A measurable function $f : X \to \mathbb{R}$ represents an *observable* of the physical system, that is a quantity that can be measured, for example velocity, position, temperature and so on. The value $f(x)$ is the measurement of the observable f that one gets when the system is in the state x. Time evolution of the system, if measured in discrete time units, is given by a transformation $T : X \to X$, so that if $x \in X$ is the initial state of the system, then $T(x)$ is the state of the system after one time unit. If the physical system is in equilibrium, the time evolution T is a measure-preserving transformation.

In order to measure a physical quantity, one usually repeats measurements in time and consider their average. If $x \in X$ is the initial state, the measurements of the observable $f : X \to \mathbb{R}$ at successive time units are given by $f(x), f(T(x)), \dots, f(T^k(x)), \dots$. Thus, the *average* of the first n measurements is given by

$$\frac{\sum_{k=0}^{n-1} f(T^k x)}{n} \qquad \text{(time average).}$$

This quantity is called *time average* of the observable f after time n.

On the other hand, the *space average* of the observable f is simply

$$\int f \mathrm{d}\mu \qquad \text{(space average).}$$

In physics one would like to know the *space average* of the observable with respect to the invariant measure, but since experimentally one computes easily time averages (just by repeating measurements of the system at successive instant of times), it is natural to ask whether (and hope that) long time averages give a good approximation of the space average. Boltzmann's conjectured the following:

Boltzmann Ergodic Hypothesis: for almost every initial state $x \in X$ the time averages of any observable f converge as time tends to infinity to the space average of f.

Unfortunately, after many efforts to prove this general form of the Boltzmann Ergodic Hypothesis, it turned out that the conclusion is *not true* in general, for any measure-preserving transformation T. On the other hand, *under the assumption that T is ergodic*, the conclusion of the Boltzmann Ergodic Hypothesis holds and this is exactly the content of Birkhoff Ergodic Theorem for ergodic transformations. Finding the right condition under which the Hypothesis holds motivated the definition of ergodicity and gave birth to the study of Ergodic theory.

Two versions of Birkhoff Ergodic Theorem

The first formulation of Birkhoff Ergodic Theorem gives a result which is weaker than the Ergodic Hypothesis, but holds in general for *any measure preserving transformation* that preserves a *finite* measure.

Theorem 3.8.1 (Birkhoff Ergodic Theorem for measure preserving transformations). *Let $(X, \mathscr{A}, \mu)$ be a* finite *measured space. Let $T : X \to X$ be* measure-preserving *transformation. For any $f \in L^1(X, \mathscr{A}, \mu)$, the following limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x))$$

exists for μ-almost every $x \in X$. *Moreover, if, for the x for which the limit exists we call*

$$\overline{f}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x))$$

the function $\overline{f}$ (which is defined almost everywhere) is invariant*, that is*

$$\overline{f} \circ T = \overline{f} \qquad \textit{for } \mu - \textit{almost every } x \in X,$$

and furthermore

$$\int \overline{f} \mathrm{d}\mu = \int f \mathrm{d}\mu.$$

Let us stress again that this theorem, as Poincaré Recurrence, follows simply from preserving a finite measure. We will not prove the theorem here.

[If you are interested, a proof can be found in the lecture notes on Ergodic Theory by *Omri Sarig* (see section 2.2 in Chapter 2) which are available online: http://www.wisdom.weizmann.ac.il/ sarigo/506/ErgodicNotes.pdf]

The following version of Birkhoff Ergodic Theorem for ergodic transformations is simply a Corollary of this general Birkhoff Ergodic Theorem:

Theorem 3.8.2 (**Birkhoff Ergodic Theorem for ergodic transformations**). *Let* $(X, \mathscr{A}, \mu)$ *be a* probability *space. Let* $T : X \to X$ *be an* ergodic *measure-preserving transformation. For any* $f \in L^1(X, \mathscr{A}, \mu)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = \int f \mathrm{d}\mu \qquad \text{for } \mu-\text{almost every } x \in X.$$

Proof. By Birkhoff Ergodic Theorem for measure preserving transformations for $\mu-$almost every x the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = \overline{f}(x)$$

exists and defines a function $\overline{f}$ such that $\overline{f} \circ T = \overline{f}$ almost everywhere. Since T is ergodic, every function which is invariant almost everywhere is constant almost everywhere. In particular, $\overline{f}$ is constant almost everywhere. If c is the value of this constant, since μ is a probability measure

$$\int \overline{f} \mathrm{d}\mu = c \cdot \mu(X) = c \cdot 1 = c,$$

but since the ergodic theorem for measure preserving transformations also gives that $\int \overline{f} \mathrm{d}\mu = \int f \mathrm{d}\mu$, we conclude that $\int f \mathrm{d}\mu = c$. Thus, for almost every $x \in X$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = c = \int f \mathrm{d}\mu,$$

which is the conclusion we were looking for. □

Extra: the proof of Birkhoff pointwise ergodic theorem

In this section (which is an Extra) we present a proof of the Birkhoff ergodic theorem for measure-preserving transformations (Theorem 3.8.1). There are many proofs, the most classical ones make use of the *maximal inequality*, which is a quite standard tool in harmonic analyses. We chose here to give a proof which is quite simple and short, even though if subtle. It is originally due to Kamae. We follow here quite closely the presentation in Sarig's lecture notes.

Proof. Let us first remark that we can assume without loss of generality that f is a *non-negative* function, i.e. $f \geq 0$, since otherwise we can write $f = f^+ - f^-$ where $f^\pm \geq 0$ are respectively the *positive* and *negative* part of f and prove separately the result for each (then combine them by linearity).

Let us also assume for now that f is *bounded*, i.e. that there exists a constant $M > 0$ such that $\sup_{x\in X} f(x) \leq M$. The last step, *Step 4*, will explain how to remove this assumption.

We will break the proof into several steps.

Step 1. Let us first of all show

□

to be added

3.9 Von Neumann Mean Ergodic Theorem

The *sums along the orbit* and the *time averages* which appear in Birkhoff ergodic theorem, namely

$$\sum_{k=0}^{n-1} f(T^k(x)), \qquad \frac{1}{n}\sum_{k=0}^{n-1} f(T^k(x)),$$

are know respectively as *Birhoff sums* (or *ergodic sums*) and (Birkhoff) *ergodic averages* (of the function f, along the orbits of the transformation T).

Consider Birkhoff averages as *functions* and introduce the notation

$$A_n(f) := \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k, \qquad n \in \mathbb{N}.$$

Here the symbol A_n can be thought as an *averaging* operator, that to the function f associate the ergodic sum function $A_n(f)$.

The Birkhoff ergodic theorem states that the sequence of functions $(A_n(f))_{n\in\mathbb{N}}$ converge *pointwise μ-almost everywhere* when the observable $f \in L^1(\mu)$. More in general, one can ask if other type of convergence of the sequence of functions $A_n(f), n \in \mathbb{N}$ converge *almost every where* for functions in $L^1(\mu)$.

Other type of convergence can be investigated as well, under different assumptions on the regularity of the observable. We will see now two further results on the convergence of the ergodic averages $A_n(f), n \in \mathbb{N}$, namely a result on their convergence in $L^2(\mu)$ when $f \in L^2(\mu)$ (due to von Neumann, see Theorem 3.9.1) and a result on uniform convergence for *continuous* functions (under stronger assumptions on T, see Theorem 3.11.1 in § 3.11.1).

All results about convergence of ergodic averages are called *ergodic theorems.* THe following ergodic theorem, about *mean convergence* (i.e. convergence in $L^2(\mu)$ is due to von Neumann and was proven essentially at the same time than the pointwise ergodic theorem by Birkhoff (in 1932 and 1931 respectively).

Theorem 3.9.1 (Von Neumann mean ergodic theorem). *Let $(X, \mathscr{A}, \mu)$ be a* probability *measured space. Let $T : X \to X$ be* measure-preserving *transformation. For any $f \in L^2(X, \mathscr{A}, \mu)$, the following limit*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k$$

exists *in $L^2(\mu)$ and is a function $\overline{f} \in L^2(\mu)$, that is*

$$\left\| \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k - \overline{f} \right\|_{L^2} \to 0$$

as $n \to \infty$ and $\overline{f} \circ T = \overline{f}$ in $L^2(\mu)$. Moreover, $\int \overline{f} \mathrm{d}\mu = \int f \mathrm{d}\mu$.

Reasoning as in the case of Birkhoff ergodic theorem, we immediately have the corollary.

Corollary 3.9.1 (mean ergodic theorem for ergodic transformations). *Let $(X, \mathscr{A}, \mu)$ be a* probability *space. Let $T : X \to X$ be an* ergodic *measure-preserving transformation.*

For any $f \in L^2(X, \mathscr{A}, \mu)$, $\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k$ cconverges in $L^2(\mu)$ to $\int f \mathrm{d}\mu$.

Proof. [The proof of Step 3 and 4 are Extra.] We will first consider two special class of functions in $L^2(\mu)$, for which it is very easy to verify the statement of the mean ergodic theorem, the subspace $\mathcal{I}$ of *invariant functions* and the set $\mathcal{C}$ of *coboundaries*:

$$\mathcal{I} := \{f \in L^2(\mu), \quad \text{such that } f = f \circ T\} \subset L^2(\mu);$$
$$\mathcal{C} := \{f \in L^2(\mu), \quad \text{such that } f = g \circ T - g,\, g \in L^2(\mu)\} \subset L^2(\mu).$$

[Functions of the form $g \circ T - g$ in dynamics are called *coboundaries* and the function g is called *transfer function*; the set $\mathcal{C}$ consists more precisely of coboundaries with $L^2(\mu)$ transfer function.] We will first prove the result for $f \in \mathcal{I}$ (in *Step* 1), then for $f \in \mathcal{C}$ (see *Step* 2).

Step 1 (proof for invariant functions): Let us first consider an invariant function $f \in \mathcal{I}$. Then, since $f = f \circ T^k$ for each $k \in \mathbb{N}$,

$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k = \frac{1}{n}\sum_{k=0}^{n-1} f = f.$$

Thus the limit of the Birkhoff averages is simply f, that in this case is invariant by assumption and the conclusion is trivially true.

Step 2 (proof of coubondaries): Let us now consider $f \in \mathcal{C}$, so that we can write $f = g \circ T - g$ for some $g \in L^2(\mu)$. Then

$$\sum_{k=0}^{n-1} f \circ T^k = \sum_{k=0}^{n-1} (g \circ T - g) \circ T^k = \sum_{k=0}^{n-1} \left(g \circ T^{k+1} - g \circ T^k\right) = g \circ T^n - g$$

since the sum is telescopic. Hence,

$$\|\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k\|_{L^2} = \|\frac{g \circ T^n - g}{n}\|_{L^2} \le \frac{\|g \circ T^n\|_{L^2} + \|g\|_{L^2}}{n} = \frac{2\|g\|_{L^2}}{n}$$

which goes to zero as $n \to \infty$. Remark that 0 (being constant) is an invariant function. Moreover, since T preserves μ we have

$$\int_X (g \circ T - g) d\mu = \int_X g \circ T d\mu - \int_X g d\mu = \int_X g) d\mu - \int_X g d\mu = 0,$$

so the integral of the limit invariant function coincides with $\int f d\mu$ and the conclusion also holds in this case.

Step 3 (proof for the closure of $\mathcal{C}$): Let us consider now the $L^2(\mu)-$*closure* $\overline{\mathcal{C}}$ of the space $\mathcal{C}$ of L^2-coboundaries, i.e. the set of $f \in L^2(\mu)$ such that there exists a sequence $(f_n)_n \subset \mathcal{C}$ such that f_n converges to f in $L^2(\mu)$ as n tends to infinity.

We claim that the conclusion also holds for $f \in \overline{\mathcal{C}}$. *to be finished...*

Step 4 (decomposition of $L^2(\mu)$ functions and conclusion): We will now show that any function $f \in L^2(\mu)$ can be (uniquely) written as $f = \overline{f} + h$ where $\overline{f} \in \mathbb{I}$ and $h \in \overline{\mathcal{C}}$. This is sufficient to conclude that the theorem holds for every f, since, by combining *Step 1* and *Step 3* we have that

$$\lim_{n\to\infty} \||\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k - \overline{f}\||_{L_2} = \lim_{n\to\infty} \||\frac{1}{n}\sum_{k=0}^{n-1} \overline{f} \circ T^k + \frac{1}{n}\sum_{k=0}^{n-1} h - \overline{f}\||_{L_2} = \||\overline{f} + 0 - \overline{f}\||_{L^2} = 0,$$

i.e. the limit of the ergodic averages in $L^2(\mu)$ is $\overline{f}$ which by construction is an invariant function and by *Step* 1 and *Step 2* we also have that

$$\int f d\mu = \int (\overline{f} + h) d\mu = \int f d\mu + 0 = \int f d\mu.$$

Equivalently, we want to prove that we have an orthogonal decomposion

$$L^2(\mu) = \mathcal{I} + \overline{\mathcal{C}}^{L^2},$$

to be finished... □

3.10 Some applications of Birkhoff Ergodic Theorem

The version of Birkhoff Ergodic Theorem for ergodic transformations shows that Boltzmann's Ergodic Hypothesis is true if the time evolution is ergodic. Birkhoff ergodic Theorem has many other applications in different areas of mathematics. We will show a few consequences.

1. Frequencies of Visits. Let $(X, \mathscr{A}, \mu)$ be a *probability* space and let $T : X \to X$ be an *ergodic* measure-preserving transformation. Let $A \in \mathscr{A}$ be a measurable set of positive measure $\mu(A) > 0$. Given $x \in X$, the frequencies of visits of x to A up to time n are given by

$$\frac{Card\ \{\ 0 \leq k \leq n-1, \qquad T^k(x) \in A\ \}}{n} = \frac{1}{n}\sum_{k=0}^{n-1} \chi_A(T^k(x)),$$

as we have already seen at the beginning of Chapter 3. If we apply Birkhoff ergodic theorem to the function $f = \chi_A$, which is measurable since $A \in \mathscr{A}$ and integrable since $\int \chi_A \mathrm{d}\mu = \mu(A) \leq \mu(X) < +\infty$, we get that for almost every x

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_A(T^k(x)) = \int \chi_A \mathrm{d}\mu = \mu(A).$$

Thus, for almost every x in A the limit of the frequencies of visits exists and is equal to $\mu(A)$:

$$\lim_{n\to\infty} \frac{Card\ \{\ 0 \leq k \leq n-1, \qquad T^k(x) \in A\ \}}{n} = \mu(A) \quad \text{for } \mu-\text{a.e. } x \in X. \tag{3.32}$$

Example 3.10.1. Let R_α be an irrational rotation. We showed in §3.6 that R_α is ergodic with respect to λ. Thus, if we take as set an interval $[a, b]$, for almost every $x \in [0, 1]$ we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_{[a,b]}(R_\alpha^k(x)) = \lambda([a, b]) = b - a.$$

Notice that this result is *stronger* than Poincaré Recurrence: indeed, given a measurable set A with positive measure, it implies that μ-a.e. point $x \in A$ not only will come back to A infinitely many times, but it will return with positive frequency (equal to $\mu(A)$). Thus, it can be though of a *quantitative version* of recurrence.

If $x \in X$ is such that (3.32) holds for all $A \in \mathbb{A}$, we say that x (or its orbit $\mathscr{O}_T^+(x)$) is *equidistributed* (or *uniformely distributed*) *with respect to* (the measure) μ.

Remark 3.10.1. In the special case of the rotation, we have already proved (see Chapter 1, the proof used Weyl's criterion for equidistribution modulo one) that the conclusion of Birkhoff ergodic theorem holds *for all* initial points $x \in X$. In particular, for example, it holds for $x = 0$. Thus, since $R_\alpha^k(0) = \{k\alpha\}$ where $\{\cdot\}$ denotes the fractional part, we have

$$\lim_{n\to\infty} \frac{Card\{0 \leq k < n,\ \{k\alpha\} \in [a, b]\}}{n} = \lambda([a, b]) = b - a.$$

Hence, the sequence $(\{k\alpha\})_{k\in\mathbb{N}}$ is *equidistributed modulo one* (a notion that we saw in Chapter 1) is equivalent to $\mathscr{O}_{R_\alpha}^+(0)$ being equidistributed with respect to the Lebesgue measure. One can recover the stronger result also without using Weyl's criterion, but starting from this application of Birkhoff ergodic theorem and then using that R_α is an isometry (se Exercise (3.10.1) following).

*** Exercise 3.10.1.** Let α be irrational. Show that if

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \chi_{[a,b]}\left((R_\alpha^k(x)\right)$$

exists for almost every point $x \in [0, 1]$, then it exists *for all* points y and

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_{[a,b]}\left((R_\alpha^k(y)\right) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_{[a,b]}\left((R_\alpha^k(x)\right).$$

2. Borel Normal Numbers. Let $x \in [0,1]$ and consider its *binary expansion*, that is

$$x = \sum_{i=1}^{\infty} \frac{a_i}{2^i},$$

where the $a_i \in \{0,1\}$ are the digits of the binary expansion of x. Remark that the binary expansion is unique for *almost every*[9] $x \in X$.

Definition 3.10.1. A number $x \in [0,1]$ is called *normal in base* 2 if the frequency of occurrence of the digit 0 is the binary expansion and the frequency of occurrence of the digit 1 both exist and equal $1/2$.

Theorem 3.10.1 (**Borel theorem on normal numbers).** *Almost every* $x \in [0,1]$ *is normal in base* 2.

Proof. Let us prove the Theorem using Birkhoff ergodic theorem. Consider the doubling map $T(x) = 2x$ mod 1. We proved that T preserves the probability measure λ on $X = [0,1]$ and is ergodic with respect to λ. Recall that we showed in §1.4.2 that

$$x = \sum_{i=1}^{\infty} \frac{a_i}{2^i} \quad \Rightarrow \quad T^k(x) = \sum_{i=1}^{\infty} \frac{a_{k+i}}{2^i},$$

that is, the doubling map act as a *shift* on the digits of the binary expansion of x. Since the first digit a_1 of the expansion is clearly $a_1 = 0$ if and only if $x \in [0,1/2)$ and $a_1 = 0$ if and only if $x \in [1/2,1]$, this shows that, since a_{k+1} is the first digit of the expansion of $T^k(x)$,

$$a_{k+1} = \begin{cases} 0 & \text{iff } T^k x \in [0,1/2) \\ 1 & \text{iff } T^k x \in [1/2,1] \end{cases}$$

Thus,

$$\begin{aligned} \frac{Card\{\, 1 \le k \le n \quad a_k = 0 \,\}}{n} &= \frac{Card\{\, 0 \le k < n \quad a_{k+1} = 0 \,\}}{n} \\ &= \frac{Card\{\, 0 \le k < n \quad T^k(x) \in [0,1/2) \,\}}{n}. \end{aligned}$$

Since T is ergodic, by Birhoff ergodic theorem applied to $f = \chi_{[0,1/2)}$, for almost every $x \in [0,1]$

$$\lim_{n\to\infty} \frac{Card\{\, 1 \le k \le n \quad a_k = 0 \,\}}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_{[0,1/2)}(T^k(x)) = \lambda([0,1/2)) = 1/2,$$

thus the frequency of occurrence of 0 is $1/2$. Similarly, for almost every $x \in [0,1]$

$$\lim_{n\to\infty} \frac{Card\{1 \le k \le n \quad a_k = 1\}}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_{[1/2,1]}(T^k(x)) = \lambda([1/2,1]) = 1/2.$$

Remark that the intersection of two full measure sets has full measure (since the complement is the union of two measure zero sets, which has measure zero). We conclude that for almost every $x \in [0,1]$ the frequency of both 0 and 1 exists and equals $1/2$, thus almost every x is normal in base 2. □

Exercise 3.10.2. Consider the unit interval $[0,1]$ with the Lebesgue measure. Let $r \ge 2$ be an integer.

(a) Give a similar definition of a number which is *normal in base* r;

(b) Show that almost every $x \in [0,1]$ is normal base r;

(c) Deduce that almost every $x \in [0,1]$ is simultaneously normal with respect to any base $r = 2,3,\dots,n,\dots$.

[9] The numbers for which it is not unique are exactly the ones of the form $k/2^n$, for which one has two expansions, one with a tail of 0 and one with a tail of 1 in the digits. Numbers of this form are clearly countable and thus have Lebesgue measure zero.

3. Continued Fractions and Gauss-Kusmin distribution Let $x \in [0,1]$ and let us express it as a continued fraction $[a_0, a_1, \ldots, a_n, \ldots]$ where a_i are the entries of the CF expansion. Let us show that for almost every $x \in [0,1]$ the frequency of occurrence of the digit k as entry of the continued fraction of x is given by

$$\frac{1}{\log 2} \log\left(\frac{(k+1)^2}{k(k+2)}\right). \tag{3.33}$$

[The probability given by (3.33) of seeing the digit k is known as *Gauss-Kusmin distribution*.]

We showed in §1.7 that the entries of the continued fraction expansion of x are given by the itinerary of $\mathcal{O}^+_G(x)$ with respect to the partition $P_k = (1/(k+1), 1/k]$, that is the entry $a_i = k$ if and only if $G^i(x) \in P_k$ (see Theorem 1.7.1). Thus,

$$\frac{Card\{0 \le j < n \quad \text{such that} \quad a_j = k\}}{n} = \frac{1}{n}\sum_{j=0}^{n-1} \chi_{P_k}(G^j(x)).$$

Since G is ergodic with respect to the Gauss measure μ, for $\mu-$almost every $x \in [0,1]$ the limit of the previous quantity as $n \to \infty$ exists and is given by

$$\lim_{n\to\infty} \frac{Card\{0 \le j < n \quad \text{such that} \quad a_j = k\}}{n} = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \chi_{P_k}(G^j(x))$$

$$= \mu(P_k) = \int_{\frac{1}{k+1}}^{\frac{1}{k}} \frac{1}{\log 2}\frac{\mathrm{d}x}{1+x} = \left.\frac{\log(1+x)}{\log 2}\right|_{\frac{1}{k+1}}^{\frac{1}{k}} = \frac{1}{\log 2}\log\left(\frac{1+\frac{1}{k}}{1+\frac{1}{k+1}}\right) = \frac{1}{\log 2}\log\left(\frac{\frac{1+k}{k}}{\frac{k+2}{k+1}}\right),$$

which, simplifying, gives (3.33).

One can show that the same conclusion holds for Lebesgue a.e.$x \in X$, since if it failed for a set of λ-positive measure A, it would fail for a set of μ-positive measure, since

$$\mu(A) = \int_A \frac{1}{(1+x)\log 2} \ge \frac{1}{2\ln 2}\lambda(A) > 0.$$

[More in general, one can show that the measure μ and the measure λ have the same sets of measure zero. Measures with these property are called *absolutely continuous* with respect to each other and if a property holds for almost every point according to one such measure, it holds also for almost every point for the other.]

Exercise 3.10.3. (a) Show that the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \log(n), \qquad \text{if } x \in P_n = \left(\frac{1}{n+1}, \frac{1}{n}\right]$$

is in $L^1(\mu)$ and that

$$\int f \mathrm{d}\mu = \sum_{n=1}^{\infty} \frac{\log n}{\log 2} \log\left(\frac{(n+1)^2}{n(n+2)}\right) < +\infty;$$

(b) Show that for almost every point $x \in [0,1]$

$$\frac{1}{n}\sum_{i=0}^{n-1} \log a_i = \int f(x)\mathrm{d}\mu;$$

(c) Deduce that for almost every point $x \in [0,1]$ the geometric mean (which is the expression in (3.34)) of the entries of the CF has a limit and

$$\lim_{N\to\infty} (a_0 a_2 \ldots a_{N-1})^{\frac{1}{N}} = \prod_{n=1}^{\infty} \left(\frac{(n+1)^2}{n(n+2)}\right)^{\frac{\log n}{\log 2}}. \tag{3.34}$$

Ergodicity and Birkhoff Ergodic Theorem

The second form of Birkhoff Ergodic theorem shows that ergodicity is sufficient for Boltzmann ergodic Hypothesis to hold. It turns out that it is also necessary: if the conclusion of Birkhoff ergodic theorem holds, that is the time averages converge to the space averages for almost every point and all observables, then the transformation T has to be ergodic. We show this in the Theorem 3.10.2 below. In the same Theorem 3.10.2 we also show how Birkohff ergodic Theorem can be rephrased in terms of measures of sets (see Part (3) in Theorem 3.10.2) to give another useful characterization of ergodicity. You can think of this result as yet another application of Birkhoff ergodic theorem.

Theorem 3.10.2. *Let* $(X, \mathscr{A}, \mu)$ *be a* probability *space and* $T : X \to X$ *a measure-preserving transformation. The following are equivalent:*

(1) T *is* ergodic*;*

(2) for any $f \in L^1(X, \mathscr{A}, \mu)$ *and* μ*-almost every* $x \in X$*,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = \int f \mathrm{d}\mu; \tag{3.35}$$

(3) for any $A, B \in \mathscr{A}$*,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(T^{-k}A \cap B) = \mu(A)\mu(B). \tag{3.36}$$

Saying that $(1), (2)$ and (3) are *equivalent* means one holds if and only if any of the others hold. In particular, (1) equivalent to (2) shows that the conclusion of the second form of Birkhoff Ergodic theorem (Boltzmann ergodic Hypothesis) holds if and only if T is ergodic.

The equivalence between (1) and (3) gives another characterization of ergodicity. We defined ergodicity in terms of triviality of invariant sets ($T^{-1}(A) = A$ implies $\mu(A) = 0$ or 1) and we already saw that equivalently invariant functions are constant ($f \circ T = f$ a.e. implies f constant a.e.). Equivalently, one can define ergodicity by requiring that any two measurable sets A, B

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(T^{-k}A \cap B) = \mu(A)\mu(B).$$

(Compare this characterization with the definition of mixing in the next section §3.8 and see the comments after (3.43) in §3.8.)

Proof of Theorem 3.10.2. We will show that (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1). This will prove the equivalence.

The implication (1) $\Rightarrow$ (2) is simply the statement of Birkhoff Ergodic Theorem for ergodic transformations: if T is ergodic, the convergence of time averages to space averages stated in (3.35) holds for all $f \in L^1(\mu)$ and almost every point.

Let us show that (2) $\Rightarrow$ (3). Assume that (3.35) holds for all $f \in L^1(\mu)$ and almost every point. To show that (3) holds, take any two measurable sets $A, B \in \mathscr{A}$. Consider the characteristic function χ_A. Since $\int \chi_A \mathrm{d}\mu = \mu(A) \leq \mu(X) < \infty$, $\chi_A \in L^1(\mu)$ and we can apply (3.35) to $f = \chi_A$. Thus,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_A(T^k(x)) = \int \chi_A \mathrm{d}\mu = \mu(A), \qquad \text{for } a.e.\ x \in X.$$

Multiplying both sides by $\chi_B(x)$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_A(T^k(x))\chi_B(x) = \mu(A)\chi_B(x), \qquad \text{for } a.e.\ x \in X. \tag{3.37}$$

Recall that we showed that $\chi_A \circ T = \chi_{T^{-1}(A)}$ (see equation (3.14) in §3.4), thus $\chi_A \circ T^k = \chi_{T^{-k}(A)}$. Let us show now that

$$\chi_A \chi_B = \chi_{A\cap B}.$$

This holds since characteristic functions take only 0 or 1 as values, so the product $\chi_A\chi_B(x)$ is equal to 1 if and only if both $\chi_A(x) = 1$ and $\chi_B(x) = 1$ (otherwise, if one of the two is 0, the product is 0 also). Thus, $\chi_A\chi_B(x) = 1$ if and only if $x \in A$ and $x \in B$, which equivalently means that $x \in A \cap B$. But a function which is 1 on $A \cap B$ and 0 otherwise is exactly the characteristic function $\chi_{A\cap B}$. Thus

$$\chi_A \circ T^k \chi_B = \chi_{T^{-k}(A)}\chi_B = \chi_{T^{-k}(A)\cap B},$$

and (3.37) can be rewritten as

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_{T^{-k}(A)\cap B}(x) = \mu(A)\chi_B(x), \qquad \text{for } \ a.e. \ x \in X. \tag{3.38}$$

Let us integrate both sides of this equation:

$$\int \frac{1}{n}\sum_{k=0}^{n-1} \chi_{T^{-k}(A)\cap B}(x)\mathrm{d}\mu \;=\; \frac{1}{n}\sum_{k=0}^{n-1} \int \chi_{T^{-k}(A)\cap B}(x)\mathrm{d}\mu = \frac{1}{n}\sum_{k=0}^{n-1} \mu(T^{-k}A \cap B), \tag{3.39}$$

$$\int \mu(A)\chi_B(x)\mathrm{d}\mu \;=\; \mu(A)\int \chi_B(x)\mathrm{d}\mu = \mu(A)\mu(B). \tag{3.40}$$

Thus, the conclusion follows if we can exchange the sign of limit with the sign of integration and show that the limit of the integrals is the integral of the limits:

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mu(T^{-k}A \cap B) &= \lim_{n\to\infty} \int \frac{1}{n}\sum_{k=0}^{n-1} \chi_{T^{-k}(A)\cap B}(x)\mathrm{d}\mu \qquad \text{(by (3.39))} \\
&= \int \left(\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_{T^{-k}(A)\cap B}(x) \right) \mathrm{d}\mu \qquad \text{(if one can exchange)} \\
&= \int \mu(A)\chi_B(x)\mathrm{d}\mu \qquad \text{(by (3.38))} \\
&= \mu(A)\mu(B) \qquad \text{(by (3.40))}.
\end{aligned}$$

The step of exchanging the sign of limit with the sign of integration can be justified by using the Dominated Convergence Theorem (see the Extra 3 in §3.4). Thus, we proved (3).

Let us show that (3) $\Rightarrow$ (1). Assume that (3.36) holds for any $A, B \in \mathscr{A}$. Let us show that T is ergodic by using the definition. Let $A \in \mathscr{A}$ be an invariant set. Apply (3.36) to A taking also $B = A$, so that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mu(T^{-k}A \cap A) = \mu(A)^2. \tag{3.41}$$

Remark that since A is invariant under T, $T^{-k}(A) = A$, so that $T^{-k}A \cap A = A \cap A = A$. Since if we sum n terms equal to $\mu(A)$ and divide by n we get $\mu(A)$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mu(T^{-k}A \cap A) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mu(A) = \lim_{n\to\infty} \mu(A) = \mu(A),$$

equation (3.41) implies that $\mu(A) = \mu(A)^2$. But the only positive real numbers such that $x = x^2$ are $x = 0, 1$. Thus either $\mu(A) = 0$ or $\mu(A) = 1$. This shows that T is ergodic. $\square$

3.11 Beyond Ergodicity: Unique Ergodicity and Mixing

In this section we define two new ergodic properties, *unnique ergodicity* and (measure theoretical, or strong) *mixing*, which are both stronger than ergodicity, in the sense that they *imply* ergodicity. They are quite different strenghthenings, though. As we will see unique ergodicity can be seen as a measure theoretical strenghtening of the notion of minimality, while mixing is a quantitative (measure theoretical) version of topologial mixing (this is made precise in § **??**).

3.11.1 Unique Ergodicity and Oxtoby Ergodic Theorem

In this section, let us assume that X is at the same time a *compact* metric space (X, d) and a measurable space $(X, \mathscr{B})$ where $\mathscr{B}$ is the σ-algebra of *Borel* measurable sets. Let $T : X \to X$ be a topological dynamical system, namely assume that T is continuous.

This is the same setup of § 3.5, in which we proved that the set $\mathcal{M}_T(X)$ (and hence $\mathcal{P}_T(X)$) of Borel finite (probability) measures invariant under T (this notation was introduced in § 3.5) is *non-empty* (see Krylov-Bogolyubov Theorem 3.5.1). Thus, T is also a measure preserving transformation $T : (X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$ for any $\mu \in \mathcal{M}_T(X)$.

Definition 3.11.1. We say that a continous transformation $T : X \to X$ of a compact metric space is *uniquely ergodic* if admints a *unique* invariant *probability* measure, i.e. $\mathcal{P}_T(X) = \{\mu\}$ is a singleton.

Remark that it is necessary to request that μ is a *probability* measure to have uniqueness, since if we *rescale* the measure and condider the measure $c\mu$ for any positive $c \in \mathbb{R}$, clearly also $c\mu$ is T-invariant. Thus, even when $\mathcal{P}_T(X) = \{\mu\}$ is a point, the set of invariant measures $\mathcal{M}_T(X)$ is a *ray*, namely

$$\mathcal{M}_T(X) = \{c\mu, c \geq 0\} = \mathbb{R}^+\{\mu\}.$$

Remark 3.11.1. *If T is uniquely ergodic and μ is the unique invariant probability measure, then μ is automatically* ergodic. *Indeed, if μ had a non-trivial (from the point of view of the measure) invariant set $A \in \mathbb{B}$, then the restrictions μ_A and $\mu_{X\backslash X}$ of the measure μ to A and $X\backslash A$ respectively (defined in § **??**) would give* two *distinct invariant probability measures, thus contradicting uniqueness. Equivalently, since we already said that* ergodic *invariant measures are in one to one correspondence with* extremal points *of $\mathcal{P}_T(X)$, it is clear that when $\mathcal{P}_T(X) = \{\mu\}$ the unique point μ is extremal and hence ergodic.*

The following theorem shows that when T is uniquely ergodic a stronger form of convergence of time averages holds (than in the Birkhoff ergodic theorem for measure-preserving transformations), i.e. ergodic averages converge not only *almost-everywhere*, but at *every* point, and furthermore the convergence is uniform.

Recall that $\mathcal{C}(X)$ denotes the space of *continuous* real valued functions $f : X \to \mathbb{R}$.

Theorem 3.11.1 (Oxtoby Ergodic Theorem)**.** *Let $T : X \to X$ a continuous map of a compact metric space (X, d). Then the following are equvalent:*

(1) T is uniquely ergodic and $\mathcal{P}_T(X) = \{\mu\}$;

(2) for every $f \in \mathcal{C}(X)$ the ergodic averages $\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k$ converge for every *point to a constant c_f which depends on the function only and the convergence is* uniform*, i.e.*

$$\lim_{n\to\infty} \sup_{x\in X} \left| \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x) - c_f \right| = 0. \tag{3.42}$$

Moreover when (1) *holds the constant c_f in* (2) *is simply $c_f = \int f d\mu$.*

Proof. to be added

□

The convergence in Theorem 3.11.1 is very strong, not so much since it is uniform, but because it holds *for every point*. This is a very *rare* strenghtening of Birkhoff ergodic theorem, which normally holds only *amost everywhere*[10]. Results for *every* points are very useful, in particular for applications to number theory (while unfortunately there are many examples in ergodic theory when we know that some results holds for *amost every* point, but are not able to exhibit even a single *explicit* point for which it holds!).

Oout of all the transformations that we encountered so far, the only ones which are *uniquely ergodic* are irrational rotations:

Exercise 3.11.1. Let $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a rotation. We know that R_α preserves λ the Lebesgue measure. If $\alpha \notin \mathbb{Q}$, one can show that R_α is *uniquely ergodic* and the unique invariant probability measure is Lebesgue, i.e. $\mathbb{P}_{R_\alpha}(\mathbb{R}/\mathbb{Z}) = \{\lambda\}$.

The proof was essentially done in the first Chapter, where we proved *Weyl criterion* for equidistribution and used it to show that orbits of irrational rotations are uniformely distributed modulo one. The proof shows indeed that (2) of Theorem 3.11.1 holds for all (real and imaginary parts of) *exponentials* $x \mapsto e^{2\pi i k x}$, where $k \in \mathbb{Z}\backslash\{0\}$ and hence, by linearitly, for all trigonometric polynomials and, by Weiestrass theorem (see Weyl's criterion), for all continuous functions. It follows by Oxtoby theorem characterization that R_α is uniquely ergodic.

Another example of uniquely ergodic transformations is given by Furstenberg skew products (see Exercise in the Extras).

3.11.2 Mixing

The second property which strenghten the notion of ergodicity, *mixing* formalizes the intuitive concept that a transformation *mixes well* from a measure-theoretical point of view. In many examples it turns out that it is easier to prove that T is mixing than to prove that T is ergodic. Thus, showing mixing will also provide us with more examples of ergodic transformation, especially in the case of shifts.

Let $(X, \mathscr{A}, \mu)$ be a *probability* space and $T : X \to X$ a measure-preserving transformation.

Definition 3.11.2. The transformation T is *mixing* with respect to the measure μ (or simply mixing) if for any pair of measurable sets $A, B \in \mathscr{A}$

$$\lim_{n\to\infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B). \tag{3.43}$$

The intuitive meaning of this definition is the following. Assume that $\mu(A) > 0$. Then we can divide both sides by $\mu(A)$ and get

$$\lim_{n\to\infty} \frac{\mu(T^{-n}A \cap B)}{\mu(A)} = \mu(B).$$

Remark that since T preserves μ, $\mu(T^{-n}(A)) = \mu(A)$ for any $n \in \mathbb{N}$, so we can rewrite

$$\frac{\mu(T^{-n}A \cap B)}{\mu(T^{-n}A)} \xrightarrow{n\to\infty} \mu(B).$$

The ratio in the left hand side is the proportion of the measure of the set $T^{-n}(A)$ which intersects B. Thus, mixing shows that the proportion of the measure of $T^{-n}(A)$ which intersects B tends, as n grows, to $\mu(B)$ (which is a number between 0 and 1). In particular, if two sets B_1 and B_2 have the same measure, the proportion of the measure $T^{-n}(A)$ in B_1 and in B_2 is the same. This formalizes the intuitive idea that the set $T^{-n}(A)$ *spreads* as n grows to become equidistributed all over the space with respect to the measure μ.

[10] There are few other instances in ergodic theory when one can prove ergodic theoretical results for *every* orbit. One celebrated such instance is given by the beautiful rigidity theorems by *Marina Ratner*, who proved equidistribution results for the orbit of *every* point in the context of unipotent flows in homogeneous dynamics. Another result of this flavour is given by the very recent work by Maryam Mirzakhani (Fields medalist 2014) with Alex Eskin and Amir Mohammadi, in the context of Teichmnueller dynamics.

If you have seen some probability theory, the mixing equation (3.43) says that the sets A and B are *asymptotically independent*, where asymptotically means that one considers backward iterates $T^{-n}(A)$ and $T^{-n}(A)$ and B tend to become independent as the time n grows.

Remark 3.11.2. Let $\mathscr{S}$ be an *algebra* of subsets which generates the whole $\sigma-$algebra $\mathscr{A}$ of measurable sets. One can prove that to show that T is mixing, it is enough to prove the mixing relation (3.43) for all sets $A, B \in \mathscr{S}$. Thus, since finite unions of the following sets form an algebra that generates the corresponding $\mathscr{A}$, it is enough to verify the mixing relation (3.43) for:

- A, B *intervals* if $X = \mathbb{R}$ or $X = I \subset \mathbb{R}$ is an interval and $\mathscr{B}$ is the the Borel $\sigma-$algebra; dyadic intervals or intervals with rational endpoints are also sufficient;
- A, B *rectangles* if $X = \mathbb{R}^2$ or $X = [0,1]^2$ and $\mathscr{B}$ is the the Borel $\sigma-$algebra; dyadic rectangles or rectangles whose vertices have rational endpoints are also sufficient;
- A, B *cylinders* if X is the shift space $X = \Sigma_N$ or Σ_N^+ or a subshift $X = \Sigma_A$ or Σ_A^+ and $\mathscr{A}$ is the $\sigma-$algebra generated by cylinders;

Mixing is a stronger property than ergodicity:

Lemma 3.11.1. *Let $(X, \mathscr{A}, \mu)$ be a* probability *space and $T : X \to X$ a measure-preserving transformation. If T is mixing with respect to μ, then T is ergodic with respect to μ.*

In particular, if we can prove that T is mixing (which sometimes turns out to be easier than to prove ergodicity) we also know that T is ergodic.

Proof of Lemma 3.11.1. Assume that T is mixing. Let us show that T is ergodic by using the definition. Let $A \in \mathscr{A}$ be an invariant set. Applying the mixing relation (3.43) to A and taking also $B = A$, we get

$$\lim_{n\to\infty} \mu(T^{-n}A \cap A) = \mu(A)\mu(A) = \mu(A)^2.$$

Since $T^{-1}(A) = A$ by invariance of A, we also have $T^{-n}(A) = A$ for any $n \in \mathbb{N}$ and $T^{-n}A \cap A = A$. Thus,

$$\mu(A)^2 = \lim_{n\to\infty} \mu(T^{-n}A \cap A) = \lim_{n\to\infty} \mu(A) = \mu(A).$$

Since the only positive real numbers such that $x = x^2$ are $x = 0, 1$, either $\mu(A) = 0$ or $\mu(A) = 1$. This shows that T is ergodic. $\square$

Let us recall that in §3.7 we showed that T is ergodic if and only if for any $A, B \in \mathscr{A}$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(T^{-k}A \cap B) = \mu(A)\mu(B). \tag{3.44}$$

Thus, if T is ergodic and $\mu(A) > 0$, using as before that by invariance $\mu(A) = \mu(T^{-k}A)$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{\mu(T^{-k}A \cap B)}{\mu(T^{-n}A)} = \mu(B),$$

which shows that the *average* of measure of the proportion of $T^{-k}(A)$ which intersects B as $0 \le k < n$ tends, as n grows, to $\mu(B)$. Thus, ergodicity can be seen as a *mixing in average* property, which is a weaker requirement than mixing.

One can use this characterization of ergodicity to show that mixing implies ergodicity:

Exercise 3.11.2. (a) Show that if $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers,

$$\lim_{n\to\infty} a_n = L \quad \Rightarrow \quad \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} a_k = L.$$

[*Hint*: Use the definition of limit. For any $\epsilon > 0$, split the sum in two parts, one corresponding to indexes k for which a_k is ϵ-close to L and the other made by finitely many a_k.]

(b) Use Part (a) to give an alternative proof that if T is mixing, then T is ergodic.

Exercise 3.11.3. Consider the probability space $(\mathbb{R}/\mathbb{Z}, \mathscr{A}, \lambda)$ and for $\alpha \in [0,1)$. Show that the irrational rotation given by $T(x) = x + \alpha \mod 1$ is not mixing with respect to λ.

Let us show two examples of mixing transformations. The first, the doubling map, is more geometric and provides a visual example of how sets can *spread uniformly* in space. The second example is the shift space with the Bernoulli measure, that we define below. In the next section, §3.9, we will construct many more measures on subshifts which are invariant under the shift map and prove that they are mixing. In all these examples related to shift spaces proving mixing turns out to be easier than proving directly ergodicity.

Example 3.11.1. [**The doubling map is mixing**] Let $T(x) = 2x \mod 1$ be the doubling map on $([0,1], \mathscr{A}, \lambda)$. Let us show that T is *mixing* with respect to the invariant measure λ. By Remark 3.11.2, it is enough to check the mixing relation (3.43) for A, B intervals. Furthermore, since also finite unions of *dyadic* intervals form an algebra and generate all Borel sets $\mathscr{A}$, it is enough to check it when A, B are *dyadic intervals*. Let us hence assume that

$$A = \left[\frac{k}{2^i}, \frac{k+1}{2^i}\right], \qquad B = \left[\frac{l}{2^j}, \frac{l+1}{2^j}\right], \qquad \text{for some } i, j \in \mathbb{N},\ 0 \le k < 2^i,\ 0 \le l < 2^j.$$

Since

$$T^{-1}(A) = \left[\frac{k}{2^{i+1}}, \frac{k+1}{2^{i+1}}\right] \cup \left[\frac{k}{2^{i+1}} + \frac{1}{2}, \frac{k+1}{2^{i+1}} + \frac{1}{2}\right]$$

the preimage $T^{-1}(A)$ consists of 2 intervals of length $1/2^{i+1}$ spaced by $1/2$. Similarly, by induction one can prove that $T^{-n}(A)$ consists of 2^n intervals of length $1/2^{i+n}$ spaced by $1/2^n$.

Thus, if $n > j$, the dyadic intervals in $T^{-n}(A)$ intersect B and the intersection $T^{-n}(A) \cap B$ consists of the intervals of $T^{-n}(A)$ which are contained in B. The number of intervals in $T^{-n}(A)$ that intersect B is given by the length of B divided by the spacing, thus the number of intervals in $T^{-n}(A) \cap B$ is

$$\frac{\lambda(B)}{\text{spacing}} = \frac{1/2^j}{1/2^n} = 2^{n-j}.$$

Since each interval has length $1/2^{i+n}$,

$$\lambda(T^{-n}(A) \cap B) = \frac{1}{2^{i+n}} 2^{n-j} = \frac{1}{2^i}\frac{1}{2^j} = \lambda(A)\lambda(B).$$

This shows that T is mixing. Remark that here there is no need to take a limit, since as soon as $n > j$, the measure of $T^{-n}(A) \cap B$ is exactly equal to the product of the measures.

Bernoulli measures on full shifts.

Let $\Sigma_N^+ = \{1, \ldots, N\}^{\mathbb{N}}$ be the full one-sided shift space and $\sigma : \Sigma_N^+ \to \Sigma_N^+$ the full one-sided shift map, that, we recall, acts like a shift on one-sided sequences:

$$\sigma((x_i)_{i=0}^{+\infty}) = (x_{i+1})_{i=0}^{+\infty}.$$

Recall that a *cylinder* in Σ_N^+ is a set of the form

$$C_n(a_0,\dots,a_n) = \{ \underline{x} = (x_i)_{i=0}^{+\infty} \in \Sigma_N^+, \text{ such that } x_i = a_i \quad \text{for all } 0 \leq i \leq n \}.$$

Let $\mathscr{A}$ be the $\sigma-$algebra generated by cylinders[11]. Let us define a class of measures called *Bernoulli measures* on the measurable space $(\Sigma_N^+, \mathscr{A})$.

Let $\underline{p} = (p_1,\dots,p_N)$ be a *probability vector*, that is a vector such that

$$0 \leq p_i \leq 1 \quad \text{for any } 1 \leq i \leq N \quad \text{and} \quad \sum_{i=1}^{N} p_i = 1.$$

For example, $(\frac{1}{N}, \frac{1}{N}, \dots, \frac{1}{N})$ is a probability vector.

Definition 3.11.3. The *Bernoulli measure* given by the probability vector $\underline{p} = (p_1,\dots,p_N)$ is the measure $\mu_{\underline{p}}$ which assigns to each cylinder $C_n(a_0,\dots,a_n)$ the measure

$$\mu_{\underline{p}}(C_n(a_0,\dots,a_n)) = p_{a_0} p_{a_1} \dots p_{a_n}.$$

By the Extension Theorem, since we defined μ on all cylinders, this automatically defines a measure on the whole $\sigma-$algebra $\mathscr{A}$ generated by cylinders.

Notice that the cylinder $C_1(i)$ consists of all sequences whose first digit is i. Since its measure is $\mu_{\underline{p}}(C_1(i)) = p_i$, p_i can be thought of as the *probability of occurrence of the digit i*. Similarly, you can think of the measure $\mu_{\underline{p}}$ of a cylinder as the probability of seeing the digit block $a_0,\dots,a_n$. Both these probabilities *do not depend on the position* in which the digit i or the block $a_0,\dots,a_n$ occurr:

Exercise 3.11.4. Show that for any $1 \leq i,j \leq N$ and any integers $k_1 \neq k_2$ the probability of seeing the digit i in position x_{k_1} and the digit j in another position x_{k_2} is given by the product $p_i p_j$.
[*Hint:* write the set of sequences $\underline{x}$ such that $x_{k_1} = i$ and $x_{k_2} = j$ as a union of cylinders and apply the definition of the measure.]

Proposition 3. *Let $\sigma : \Sigma_N^+ \to \Sigma_N^+$ be the shift and $\mu_{\underline{p}}$ the Bernoulli measure on $(\Sigma_N^+, \mathscr{A})$ given by the probability vector $\underline{p}$:*

(i) the measure $\mu_{\underline{p}}$ is invariant under the shift σ;

(ii) the shift σ is mixing *with respect to the Bernoulli measure $\mu_{\underline{p}}$.*

Corollary 3.11.1. *The shift $\sigma : \Sigma_N^+ \to \Sigma_N^+$ is* ergodic *with respect to the Bernoulli measure $\mu_{\underline{p}}$.*

Proof of Proposition 3. Let us prove (1). To show that σ preserves $\mu_{\underline{p}}$, let verify the relation $\mu_{\underline{p}}(\sigma^{-1}(A)) = \mu_{\underline{p}}(A)$ when the measurable set A is a cylinder. Consider the cylinder $C_n(a_0,\dots,a_n)$. Given $\underline{x} = (x_i)_{i=1}^{\infty} \in \Sigma_N^+$

$$\begin{aligned} &\underline{x} \in \sigma^{-1}(C_n(a_0,\dots,a_n)) \quad \Leftrightarrow \quad \sigma(\underline{x}) \in (C_n(a_0,\dots,a_n)) \quad \Leftrightarrow \\ &\sigma(\underline{x})_i = a_i \quad \text{for } \ 0 \leq i \leq n \quad \Leftrightarrow \quad (\underline{x})_{i+1} = a_i \quad \text{for } \ 0 \leq i \leq n. \end{aligned}$$

Thus, if $\underline{x} \in \sigma^{-1}(C_n(a_0,\dots,a_n))$, $x_1 = a_0, \dots, x_{n+1} = a_n$, while x_0 can be any digit in $\{1,\dots,N\}$. We can express this condition as a union of cylinders of length $n+1$:

$$\sigma^{-1}(C_n(a_0,\dots,a_n)) = \bigcup_{j=1}^{N} C_{n+1}(j, a_0, \dots, a_n).$$

[11] We saw in §2.7 that $\Sigma_N^+ = \{1,\dots,N\}^{\mathbb{N}}$ is also a metric space with the distance d_ρ. Thus, there is also a Borel $\sigma-$algebra on Σ_N^+ generated by open sets. One can see that the $\sigma-$algebra $\mathscr{A}$ generated by cylinders is the same than the Borel $\sigma-$algebra. This follows from the remark that if ρ is sufficiently large, cylinders are open sets in the metric d_ρ.

Hence, by definition of the Bernoulli measure on cylinders and additivity of a measure we get

$$\begin{aligned}\mu_{\underline{p}}\left(\sigma^{-1}(C_n(a_0,\dots,a_n))\right) &= \mu_{\underline{p}}\left(\bigcup_{j=1}^{N} C_{n+1}(j,a_0,\dots,a_n)\right)\\ &= \sum_{j=1}^{N}\mu_{\underline{p}}\left(C_{n+1}(j,a_0,\dots,a_n)\right) = \sum_{j=1}^{N} p_j p_{a_0}p_{a_1}\cdots p_{a_n}\\ &= \left(\sum_{j=1}^{N} p_j\right) p_{a_0}p_{a_1}\cdots p_{a_n} = p_{a_0}p_{a_1}\cdots p_{a_n} \qquad \left(\text{since } \sum_{j=1}^{N} p_j = 1\right)\\ &= \mu_{\underline{p}}\left(C_n(a_0,\dots,a_n)\right).\end{aligned}$$

This shows that $\mu_{\underline{p}}(\sigma^{-1}(A)) = \mu_{\underline{p}}(A)$ holds when A is a cylinder and by the Extension theorem this show that it holds for all $A \in \mathscr{A}$, so $\mu_{\underline{p}}$ is invariant under the shift.

Let us prove (2). By the Remark 3.11.2, it is enough to verify the mixing relation

$$\lim_{n\to\infty}\mu_{\underline{p}}(\sigma^{-n}A\cap B) = \mu_{\underline{p}}(A)\mu_{\underline{p}}(B)$$

when A, B are cylinders. Let

$$A = C_k(a_0,\dots,a_k), \qquad B = C_l(b_0,\dots,b_l).$$

If $n \geq l+1$ and $\underline{x} \in \sigma^{-n}(A)\cap B$, $\underline{x}$ has the following form:

$$\underline{x} = b_0, b_1, \dots, b_l, x_{l+1}, \dots, x_{n-1}, a_0, \dots, a_k, \dots$$

where the $n-1-l$ entries $x_{l+1},\dots,x_{n-1}$ can be any digit in $\{1,\dots,N\}$. Indeed, since the initial block of digits is $b_0, b_1, \dots, b_l$, $\underline{x} \in C_l(b_0,\dots,b_l) = B$. Since $x_{n+i} = a_i$ for $0 \leq i \leq k$, after shifting $\underline{x}$ to the left n times we get

$$\sigma^n(\underline{x}) = (x_{n+i})_{i\in\mathbb{N}} = a_0, \dots, a_k, \dots$$

so $\sigma^n(\underline{x}) \in C_k(a_0,\dots,a_k) = A$ and $\underline{x} \in \sigma^{-n}(A)$.

Thus, we can see the set $\sigma^{-n}(A)\cap B = \sigma^{-n}(C_k(a_0,\dots,a_k))\cap C_l(b_0,\dots,b_l)$ for $n \geq l+1$ as a union of cylinders of length $n+k$, each obtained by fixing one of the possible choices of $x_{l+1},\dots,x_{n-1}$:

$$\sigma^{-n}(A)\cap B = \bigcup_{1\leq x_{l+1},\dots,x_{n-1}\leq N} C_{n+k}(b_0,b_1,\dots,b_l,x_{l+1},\dots,x_{n-1},a_0,\dots,a_k).$$

Thus, by definition of the Bernoulli measure on cylinders and additivity of a measure we get

$$\begin{aligned}\mu_{\underline{p}}\left(\sigma^{-n}(A)\cap B\right) &= \sum_{1\leq x_{l+1},\dots,x_{n-1}\leq N} p_{b_0}p_{b_1}\cdots p_{b_l}p_{x_{l+1}}\cdots p_{x_{n-1}}p_{a_0}\cdots p_{a_k}\\ &= p_{b_0}p_{b_1}\cdots p_{b_l}\; p_{a_0}\cdots p_{a_k}\sum_{x_{l+1}=1}^{N}\sum_{x_{l+2}=1}^{N}\cdots\sum_{x_{n-1}=1}^{N} p_{x_{l+1}}\cdots p_{x_{n-1}}\\ &= (p_{b_0}p_{b_1}\cdots p_{b_l})(p_{a_0}\cdots p_{a_k}) = \mu_{\underline{p}}(A)\mu_{\underline{p}}(B).\end{aligned}$$

where we used that $\sum_{x_i=1}^{N} p_{x_i} = 1$ for each $i = l+1, \dots n-1$ since $\underline{p}$ is a probability vector. This concludes the proof of mixing. □

Similar definitions can be given for the full *bi-sided shift.* Let $\Sigma_N = \{1, \dots, N\}^{\mathbb{Z}}$ and $\sigma : \Sigma_N \to \Sigma_N$ be the shift map, whose action on Σ_N is given by

$$\sigma((x_i)_{i=-\infty}^{+\infty}) = (x_{i+1})_{i=-\infty}^{+\infty}.$$

Recall that a *cylinder* in Σ_N is a set of the form

$$C_{-m,n}(a_{-m}, \dots, a_n) = \{ \ \underline{x} = (x_i)_{i=-\infty}^{+\infty} \in \Sigma_N, \ \text{ such that } \ x_i = a_i \quad \text{for all } -m \le i \le n \ \}.$$

Let $\mathscr{A}$ be the $\sigma-$algebra generated by cylinders[12] in Σ_N. *Bernoulli measures* on the measurable space $(\Sigma_N, \mathscr{B})$ are defined similarly:

Definition 3.11.4. The *Bernoulli measure* given by the probability vector $\underline{p} = (p_1, \dots, p_N)$ is the measure $\mu_{\underline{p}}$ which assigns to each cylinder $C_n(a_{-m}, \dots, a_n)$ the measure

$$\mu_{\underline{p}}\left(C_{-m,n}(a_{-m}, \dots, a_n)\right) = p_{a_{-m}} p_{a_{-m+1}} \cdots p_{a_n}.$$

By the Extension Theorem, since we defined μ on all cylinders, this automatically defines a measure on the whole $\sigma-$algebra $\mathscr{B}$ generated by cylinders.

Showing invariance in this case is even easier, since the preimage of a cylinder is again just one cylinder.

Exercise 3.11.5. Let $\sigma : \Sigma_N \to \Sigma_N$ be the two-sided shift. Show that the Bernoulli measure $\mu_{\underline{p}}$ on $(\Sigma_N, \mathscr{B})$ given by the probability vector $\underline{p}$ is invariant under the shift σ and that σ is *mixing* with respect to $\mu_{\underline{p}}$.

3.12 Markov measures on Markov chains

In this section we will define a large class of measures on subshift spaces. We first recall the relevant definitions of shift spaces and topological Markov chains from Chapter 2. Let Σ_N be the full one-sided and let Σ_N^+ be the full two-sided shift space respectively. Let A be an $N \times N$ transition matrix. Recall that in Chapter 2, §2.7, we defined the subshift spaces associated to the matrix A as respectively as

$$\Sigma_A^+ = \{(a_i)_{i=0}^{+\infty} \in \Sigma_N^+, \quad A_{a_i a_{i+1}} = 1 \quad \text{for all } i \in \mathbb{N}\},$$
$$\Sigma_A = \{(a_i)_{i=-\infty}^{+\infty} \in \Sigma_N, \quad A_{a_i a_{i+1}} = 1 \quad \text{for all } i \in \mathbb{Z}\}.$$

Recall also that admissible cylinders (that is, non-empty cylinders) in Σ_A^+ are cylinders of the form

$$C_n(a_0, \dots, a_n) = \{ \ \underline{x} = (x_i)_{i=0}^{+\infty} \in \Sigma_N^+, \ \text{ such that } \ x_i = a_i \quad \text{for all } 0 \le i \le n, \\ A_{a_i, a_{i+1}} = 1 \text{ for all } 0 \le i < n \ \},$$

and admissible cylinders (that is, non-empty cylinders) in Σ_A are cylinders of the form

$$C_{-m,n}(a_{-m}, \dots, a_n) = \{ \ \underline{x} = (x_i)_{i=-\infty}^{+\infty} \in \Sigma_N, \ \text{ such that } \ x_i = a_i \quad \text{for all } -m \le i \le n, \\ A_{a_i, a_{i+1}} = 1 \text{ for all } -m \le i < n \ \}.$$

We will consider as measurable spaces $(\Sigma_A^+, \mathscr{B})$ and $(\Sigma_A, \mathscr{B})$ where the $\sigma-$algebras $\mathscr{B}$ are generated by the corresponding admissible cylinders.

Recall also that the restriction of the full shifts σ and σ to Σ_A^+ and Σ_A respectively are the maps

$$\sigma : \Sigma_A^+ \to \Sigma_A^+, \qquad \sigma : \Sigma_A^+ \to \Sigma_A^+,$$

which are called *topological Markov chains.*

Let us now define a large class of measures on $(\Sigma_A^+, \mathscr{B})$ and $(\Sigma_A, \mathscr{B})$, known as *Markov measures.* All these measures are preserved by a topological Markov chains and we will prove that topological Markov chains are mixing with respect to any of them.

[12] Again, this coincide with the Borel $\sigma-$algebra generated by open sets with respect to the metric d_ρ since if ρ is sufficiently large, as we saw in §2.7, cylinders are open sets.

Definition 3.12.1. An $N \times N$ matrix P is called *stochastic* if $P \geq 0$, that is all entries $P_{ij} \geq 0$ for all $1 \leq i, j \leq N$ and

$$\sum_{j=1}^{N} P_{ij} = 1, \qquad \text{for all } 1 \leq i \leq N,$$

that is the sum of entries of each *row* of P is one.
We say that the stochastic matrix P is is *compatible* with the transition matrix A if

$$P_{ij} > 0 \qquad \Leftrightarrow \qquad A_{ij} = 1.$$

Example 3.12.1. The following matrix P is an example of a stochastic matrix compatible with the transition matrix A below:

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Remark that the sum of each row is equal to one (while this is not the case for the sum of columns) and that the entries which are non-zero are exactly the ones for which $A_{ij} = 1$.

Exercise 3.12.1. Show that if P is stochastic if and only if the vector $(1, 1, \ldots, 1) \in \mathbb{R}^N$ is a right-eigenvector.

Recall that:

- $P \geq 0$ is *irreducible* if for any $i, j \in \{1, \ldots, N\}$ there exists n, possibly dependent on i, j, such that $P^n_{ij} > 0$;
- $P \geq 0$ is *aperiodic* (or *primitive*) if there exists $n \in \mathbb{N}$ such that $P^n > 0$, that is $P^n_{ij} > 0$ for any $i, j \in \{1, \ldots, N\}$.

In order to define Markov measures, we need the following result, that we will prove in the next section:

Theorem 3.12.1 (Perron Frobenius for Stochastic Matrices). *If $P \geq 0$ is a* stochastic matrix *and P is* irreducible*, then 1 is a simple eigenvalue which has a unique (up to scaling) left eigenvector $(p_1, \ldots p_N)$ which is a positive probability vector. That is a vector $\underline{p}$ such that $p_i > 0$ for each $1 \leq i \leq N$, $\sum_{i=1}^{N} p_1 = 1$ and*

$$(p_1, \ldots, p_N)P = (p_1, \ldots, p_N) \quad \Leftrightarrow \quad \sum_{i=1}^{N} p_i P_{ij} = p_j, \quad \text{for all } 1 \leq j \leq N. \tag{3.45}$$

Moreover, if P is aperiodic *(sometimes called primitive), then*

$$\lim_{n \to \infty} P^n_{ij} = p_j, \qquad \text{for all } 1 \leq i, j \leq N.$$

Thus, the matrices P^n are converging to a matrix with all *rows equal* and equal to $\underline{p}$:

$$\begin{pmatrix} & & \\ & P^n_{ij} & \\ & & \end{pmatrix} \xrightarrow{n \to \infty} \begin{pmatrix} p_1 & \ldots & p_N \\ \vdots & & \vdots \\ p_1 & \ldots & p_N \end{pmatrix}.$$

This latter result is also known as *ergodic theorem for Markov chains.* Let us use Theorem 3.12.1 to define Markov measures and to prove that they are mixing.

If P is stochastic and irreducible, by the above Theorem 3.12.1 there exists a unique positive left eigenvector $\underline{p} = (p_1, \ldots, p_N)$ such that (3.45) holds.

Definition 3.12.2. Let P be an irreducible stochastic matrix compatible with A and let $\underline{p}$ be the corresponding probability vector $\underline{p}$ such that $\underline{p} = \underline{p}P$. The *Markov measure* μ_P associated to P is the measure defined on each cylinder as

$$\mu_P(C_n(a_0, \ldots, a_n)) = p_{a_0} P_{a_0 a_1} \cdots P_{a_{n-1} a_n}.$$

One can check that this definition satisfies the assumptions of the Extension theorem and thus defines a measure on $(\Sigma_A^+, \mathscr{B})$ on the $\sigma-$algebra $\mathscr{B}$ generated by cylinders.

You can think of the measure $\mu_P(C_n(a_0, \ldots, a_n))$ as the probability of seeing the block $a_0, \ldots, a_n$. The entry p_i of the vector $\underline{p}$ gives the probability that the first digit is i and the stochastic matrix P gives *transition probabilities*: the entry P_{ij} gives the probability that the digit i is followed by the digit j. Thus, the probability of seeing $a_0, \ldots, a_n$ is the product of the probability p_{a_0} of starting with the digit a_0 times the probabilities of all transitions from a_i to a_{i+1} for $0 \leq i \leq n-1$, given by $P_{a_i a_{i+1}}$ for $0 \leq i \leq n-1$.

One can show that any Markov measure μ_P is a is a probability measure. Furthermore, the following remark follows from the fact that P is compatible with A.

Remark 3.12.1. Since P is compatible with A, we have that $\mu_P(C_n(a_0, \ldots, a_n)) = 0$ if and only the cylinder $C_n(a_0, \ldots, a_n)$ is empty, i.e. if $a_0, \ldots, a_n$ is not admissible, i.e. $A_{a_i a_{i+1}} = 0$ for some $0 \leq i < n$.

Exercise 3.12.2. Let A be a $N \times N$ transition matrix and P an $N \times N$ irreducible stochastic matrix compatible with A. Let μ_P be the associated Markov measure.

(a) Prove that $\mu_P(\Sigma_A^+) = 1$.

(b) Prove Remark 3.12.1.

Proposition 4. *Let P be an irreducible stochastic matrix compatible with A and let μ_P be the associated Markov measure. Then:*

(1) the Markov measure μ_P is invariant *under the Markov chain $\sigma : \Sigma_A^+ \to \Sigma_A^+$;*

(2) if P is in addition aperiodic*, the Markov chain $\sigma : \Sigma_A^+ \to \Sigma_A^+$ is* mixing *with respect to μ_P.*

Proof. To check that σ preserves μ_P, let us show that the relation

$$\mu_P(\sigma^{-1}(A)) = \mu_P(A) \tag{3.46}$$

holds for all A which are cylinders. This, by the Extension theorem, is enough to deduce that the relation (3.46) holds for all A in the $\sigma-$algebra $\mathscr{B}$ generated by cylinders. Let A be the cylinder

$$A = C_n(a_0, \ldots, a_n).$$

As we already saw, the preimage of A can be written as union of disjoint cylinders

$$\sigma^{-1}(C_n(a_0, \ldots, a_n)) = \bigcup_{j=1}^{N} C_{n+1}(j, a_0, \ldots, a_n)$$

[where here, possibly, some cylinders could be empty. This happens when $A_{j a_0} = 0$, so that the transition from v_j to v_{a_0} is not allowed].

The measure of the preimage, by additivity of μ_P is hence

$$\mu_P\left(\sigma^{-1}(C_n(a_0, \ldots, a_n))\right) = \sum_{j=1}^{N} \mu_P(C_{n+1}(j, a_0, \ldots, a_n)) = \sum_{j=1}^{N} p_j P_{j a_0} P_{a_0 a_1} \cdots P_{a_{n-1} a_n}.$$

Since $\underline{p}$ is a left eigenvector,

$$\underline{p}P = \underline{p} \quad \Rightarrow \quad \sum_{j=1}^{N} p_j P_{j a_0} = p_{a_0}.$$

Thus,

$$\begin{aligned}\mu_P \sigma^{-1}(C_n(a_0, \ldots, a_n)) &= \left(\sum_{j=1}^{N} p_j P_{j a_0} \right) P_{a_0 a_1} \cdots P_{a_{n-1} a_n} \\ &= p_{a_0} P_{a_0 a_1} \cdots P_{a_{n-1} a_n} = \mu_P(C_n(a_0, \ldots, a_n)).\end{aligned}$$

This concludes the proof of (1).

Assume that P is aperiodic. To prove that $\sigma : \Sigma_A^+ \to \Sigma_A^+$ is mixing with respect to μ_P it is enough to verify the mixing relation

$$\lim_{n\to\infty} \mu_P(\sigma^{-n}(A) \cap B) = \mu_P(A)\mu_P(B)$$

for all A, B which are admissible cylinders. Let

$$A = C_n(a_0, \ldots, a_k), \quad A_{a_i a_{i+1}} = 1, \quad 0 \le i < k; \qquad B = C_l(b_0, \ldots, b_l) \quad A_{b_i b_{i+1}} = 1, \quad 0 \le i < l.$$

As we have already seen, if $n > l$ the intersection $\sigma^{-n}(A) \cap B$ consists of all sequences $\underline{x} \in \Sigma_A^+$ of the form

$$\underline{x} = b_0, b_1, \ldots, b_l, x_{l+1}, x_{l+2}, \ldots, x_n, a_0, \ldots, a_k, \ldots$$

where $x_{l+1}, \ldots, x_n$ vary between all possible *admissible sequences* of length $n - l + 1$ connecting b_l to a_0, that is all sequences such that $A_{b_l x_{l+1}} = 1$, $A_{x_i x_{i+1}} = 1$ for all $l + 1 \le i \le n$ and $A_{x_n a_0} = 1$. We have

$$\mu_P(\sigma^{-n}(A) \cap B) = \sum_{x_{l+1},\ldots,x_n} p_{b_0} P_{b_0 b_1} \cdots P_{b_{l-1} b_l} \; P_{b_l x_{l+1}} P_{x_{l+1} x_{l+2}} \cdots P_{x_n a_0} P_{a_0 a_1} \cdots P_{a_{k-1} a_k},$$

where the sum vary among all possible admissible sequences $x_{l+1}, \ldots, x_n$. Since P is compatible with A, if $A_{x_i x_{i+1}} = 0$ then $P_{x_i x_{i+1}} = 0$. Thus, we can sum over *all* possible choices of $x_{l+1}, \ldots, x_n$ in $\{1, \ldots, N\}$ and have the same result. Thus

$$\mu_P(\sigma^{-n}(A) \cap B) = p_{b_0} P_{b_0 b_1} \cdots P_{b_{l-1} b_l} \sum_{x_{l+1}=1}^{N} \cdots \sum_{x_n=1}^{N} P_{b_l x_{l+1}} \cdots P_{x_n a_0} P_{a_0 a_1} \cdots P_{a_{k-1} a_k}.$$

Notice that the term that we gathered out of the sum is $\mu_P(B) = p_{b_0} P_{b_0 b_1} \cdots P_{b_{l-1} b_l}$. Remark also that, by definition of product of matrices,

$$P_{b_l a_0}^{n-l+1} = \sum_{x_{l+1}=1}^{N} \sum_{x_{l+2}=1}^{N} \cdots \sum_{x_n=1}^{N} P_{b_l x_{l+1}} P_{x_{l+1} x_{l+2}} \cdots P_{x_n a_0}.$$

Thus,

$$\mu_P(\sigma^{-n}(A) \cap B) = \mu_P(B) P_{b_l a_0}^{n-l+1} P_{a_0 a_1} \cdots P_{a_{k-1} a_k}.$$

Since P is aperiodic, by Theorem 3.12.1, $P_{b_l a_0}^{n-l+1} \to p_{a_0}$ as $n \to \infty$, thus

$$\begin{aligned}\lim_{n\to\infty} \mu_P(\sigma^{-n}(A) \cap B) &= \mu_P(B) \left(\lim_{n\to\infty} P^{n-l+1}(b_l a_0) \right) P_{a_0 a_1} \cdots P_{a_{k-1} a_k} \\ &= \mu_P(B) \; p_{a_0} P_{a_0 a_1} \cdots P_{a_{k-1} a_k} = \mu_P(B)\mu_P(A).\end{aligned}$$

This concludes the proof of mixing. $\square$

3.13 Perron Frobenius Theorem, the Ergodic Theorem for Markov chains and Google.

In this section we skecth the proof of the following Theorem, that we already used in the previous class §3.9 to define Markov measures and prove that they are mixing (the statement in the second part is slightly different but implies the formulation presented in §3.9, see the Corollary below).

Theorem 3.13.1. *Let $P \geq 0$ be a* stochastic matrix *and assume that P is* irreducible *and* aperiodic*. Then:*

*(1) (***Perron Frobenius theorem for stochastic matrices***) P has a unique left* positive *eigenvector (normalized as probability vector), that is a probability vector $\underline{p}$ such that $p_i > 0$ for each $1 \leq i \leq N$ and*

$$(p_1, \dots, p_N)P = (p_1, \dots, p_N) \quad \Leftrightarrow \quad \sum_{i=1}^{N} p_i P_{ij} = p_j, \quad \textit{for all } 1 \leq j \leq N.$$

*(2) (***Ergodic theorem for Markov Chains***) for any probability vector $\underline{q} \in \mathbb{R}^N$ we have*

$$\lim_{n\to\infty} \underline{q}P^n = \underline{p}, \qquad \Leftrightarrow \qquad \lim_{n\to\infty} \sum_{i=1}^{N} q_i P_{ij}^n = p_j \quad \textit{for all } 1 \leq j \leq N,$$

where $\underline{p}$ is the positive left eigenvector from Part (1)*. Moreover, $\underline{q}P^n$ converges to $\underline{p}$* exponentially fast*.*

Remark 3.13.1. *Part* (1) *of the Theorem holds more in general assuming only that P is irreducible, not necessarily aperiodic. We give here only the proof for P aperiodic since it is simpler.*

Part (2) of the Theorem 3.13.1 is used by the search engine Google (see Extra below). Moreover, Part (2) of Theorem implies as a Corollary the second part of the statement of Theorem 3.9.1:

Corollary 3.13.1. *If P is* aperiodic *(sometimes called primitive), and $\underline{p}P = \underline{p}$ is the unique positive probability right eigenvector of P,*

$$\lim_{n\to\infty} P_{ij}^n = p_j, \qquad \textit{for all } 1 \leq i, j \leq N.$$

We will prove the Corollary 3.13.1 after the proof of Theorem 3.13.1.

We will write that a vector $\underline{x} > 0$ and say that $\underline{x}$ is *positive* iff $x_i > 0$ for all $1 \leq i \leq N$ and similarly we will write $\underline{x} \geq 0$ and say that $\underline{x}$ is *non-negative* iff $x_i \geq 0$ for all $1 \leq i \leq N$.

In the proof we will use the following simple verification:

Exercise 3.13.1. If P is a stochastic matrix, for any $n \in \mathbb{N}$ also P^n is a stochastic matrix.

Proof of Theorem 3.12.1. Let P be a non-negative irreducible and aperiodic matrix and let us show that P has a unique positive left eigenvector. We can assume that $P > 0$ is strictly positive. If it is not, since it is irreducible, there exists $n \in \mathbb{N}$ such that $P^n > 0$ and we can work with P^n instead[13], since, by Exercise 3.13.1 also P^n is a stochastic matrix.

Let us use P to define a dynamical system $T : X \to X$ so that the eigenvalue equation becomes a fixed point equation for T. Let X be the subset of non negative vectors whose components add up to 1:

$$X = \{\underline{x} = (x_1, \dots, x_N) \in \mathbb{R}^N \text{ such that } \quad \underline{x} > 0, \quad \sum_{i=1}^{N} x_i = 1\}.$$

Let us denote by $\|x\|$ the following norm

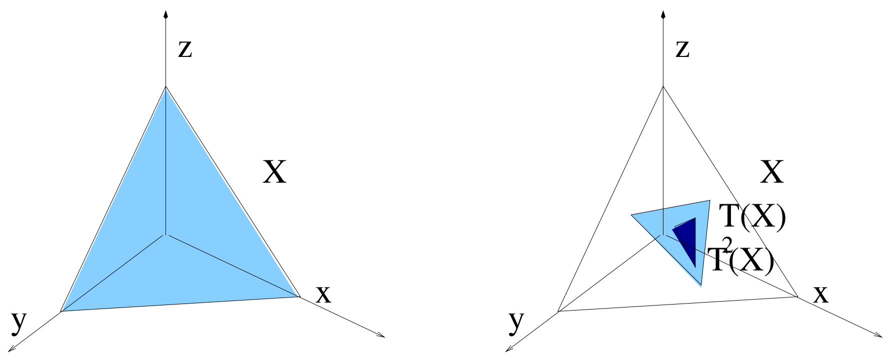
$$\|x\| = \sum_{i=1}^{N} |x_i| = \sum_{i=1}^{N} x_i \quad \text{if } \underline{x} \text{ is non-negative.}$$

[13]One can show that if the conclusions of the theorem hold for $P^n > 0$, then they also hold for P.

Thus, we can also write that X is the subset positive quadrant $\mathbb{R}_+^N$ such that

$$X = \{\underline{x} \in \mathbb{R}_+^N, \qquad \|x\| = 1\} \subset \mathbb{R}_+^N.$$

The space X is a face of a *simplex*, for example for $N = 3$ it is the face of the simplex in $\mathbb{R}_3$ shown in Figure 3.3(a).



(a) (b)

Figure 3.3: The simplex X when $N = 3$ and contraction of the map $T : X \to X$.

Define the transformation $T : X \to X$ using left multiplication by P, that is

$$T(\underline{x}) = \underline{x}P \quad \Leftrightarrow \quad T(\underline{x})_j = \sum_{i=1}^{N} x_i P_{ij}.$$

Let us check that T is well-defined, that is if $\underline{x} \in X$, then $T(\underline{x}) \in X$. Since $P > 0$ and $\underline{x} \geq 0$, $x_i P_{ij} \geq 0$ for each i, j, so also the sum over i is non-negative for any i. This shows that $T(\underline{x})_j \geq 0$ for any j. Let us compute the norm of $T(\underline{x})$: Using that P is stochastic and that $\|\underline{x}\| = 1$ we have

$$\begin{aligned}
\|T(\underline{x})\| = \sum_{j=1}^{N} T(\underline{x})_j = \sum_{j=1}^{N} \sum_{i=1}^{N} x_i P_{ij} &= \sum_{i=1}^{N} x_i \sum_{j=1}^{N} P_{ij} \\
&= \sum_{i=1}^{N} x_i \qquad \text{(since } P \text{ is stochastic)} \\
&= 1 \qquad \text{(since } \|\underline{x}\| = 1\text{)}.
\end{aligned}$$

Thus, $T(\underline{x}) \in X$ and T is well-defined. Finding a left eigenvector for P is equivalent to finding a fixed point for T, since by definition of T

$$T(\underline{p}) = \underline{p} \qquad \Leftrightarrow \qquad \underline{p}P = \underline{p}.$$

Moreover points $\underline{p} \in X$ are probability vectors since $\|\underline{p}\| = 1$. The fixed point gives a *positive* left eigenvector if and only if it is in the *interior* of X (the boundary of X consists exactly of vectors which has at least one component equal to zero).

To prove that T has a fixed point and that the fixed point is unique we will show that T *contracts* a distance. Let $d : X \times X \to \mathbb{R}^+$ be

$$d(\underline{x}, \underline{y}) = \log \frac{\max_i x_i/y_i}{\min_i x_i/y_i}, \qquad \underline{x}, \underline{y} \in X.$$

Remark that since the maximum is greater that the minimum of the rations, the argument of the logarithm is always greater or equal than 1, so d is non negative. One can show that d is symmetric and satisfies the triangle inequality. Moreover, if $d(\underline{x},\underline{y}) = 0$, all ratios x_i/y_i, $1 \leq i \leq N$ are equal, so $\underline{x}$ is a scalar multiple of $\underline{y}$. Since $\|x\| = \|y\| = 1$, $\underline{x} = \underline{y}$. Thus d is a *distance* on X. It is known as *Hilbert projective distance.*

One can show that since $P > 0$, the associated transformation T is a *strict contraction* of the Hilbert projective distance, that is there exists a constant $0 < \nu_P < 1$ such that

$$d(T(\underline{x}), T(\underline{y})) \leq \nu_P \; d(\underline{x},\underline{y}), \qquad \left(\nu_P \text{ is explicitely given by } 1 - e^{\max_{i,j,k} \frac{A_{ik}/A_{ik}}{A_{jk}/A_{jk}}}\right).$$

A transformation T which strictly contracts a distance, has a unique fixed point (this result is known as *Contraction theorem*). Indeed, one can show that the nexted sets $T^n(X)$, where $n \in \mathbb{N}$, are *nested*, that is

$$T^{n+1}(X) \subset T^n(X), \quad n \in \mathbb{N}.$$

(see Figure 3.3(a)) and since T is a contraction

$$diam_d(T^n(X)) = sup_{\underline{x},\underline{y} \in X} d(T^n(\underline{x}), T^n(\underline{y})) \xrightarrow{n\to\infty} 0.$$

Thus, the intersection $\bigcap_{n\in\mathbb{N}} T^n(X)$ is non-empty and consists of a unique point:

$$\{\underline{p}\} = \bigcap_{n\in\mathbb{N}} T^n(X).$$

One can show that the intersection point $\underline{p}$ is the unique fixed point and since $T(X)$ (and thus the intersection) is contained in the interior of X, $\underline{p}$ gives a *positive* left eigenvector.

Let us now prove Part (2). If $\underline{y}$ is any non-negative probability vector, $\underline{y} \in X$. Thus, since $\underline{p}$ is fixed and T is a strict contraction,

$$d(T^n(\underline{q}), \underline{p}) = d(T^n(\underline{q}), T^n(\underline{p})) \leq \nu_P^n d(q,p). \tag{3.47}$$

Thus, since $\nu_P < 1$, $T^n\left(\underline{q}\right)$ converges to $\underline{p}$ exponentially fast (the distance is decreasing to 0 exponentially fast) and, recalling the definition of T, this gives that

$$T^n(\underline{q}) = \underline{q}P^n \xrightarrow{n\to\infty} \underline{p}$$

and the convergence is exponential, that is the distance decreases exponentially fast, as shown by (3.47). $\square$

Let us now prove the Corollary.

Proof of Corollary 3.13.1. For each $1 \leq i \leq d$, let $\underline{e}^{(i)}$ be the vector whose i^{th} entry is equal to 1 and all other entries are zero, that is

$$e^{(1)} = \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \end{pmatrix}, e^{(2)} = \begin{pmatrix} 0 \\ 1 \\ \dots \\ 0 \end{pmatrix}, \dots, e^{(N)} = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 1 \end{pmatrix}.$$

Fix $1 \leq i \leq d$. Since $\underline{e}^{(i)}$ is a non-negative vector, Part (2) of Theorem 3.10.1 applyed to $\underline{q} = \underline{e}^{(i)}$ gives

$$\lim_{n\to\infty} \underline{e}^{(i)} P^n = \underline{p}. \tag{3.48}$$

Since $(\underline{e}^{(i)})_i = 1$ and $(\underline{e}^{(i)})_l = 0$ for all $1 \leq l \leq d$ with $l \neq i$,

$$(\underline{e}^{(i)} P^n)_j = \sum_{l=1}^{N} (\underline{e}^{(i)})_l P^n_{lj} = P^n_{ij}, \tag{3.49}$$

so combining (3.49) and (3.48)

$$\lim_{n\to\infty} P^n_{ij} = \lim_{n\to\infty} (\underline{e}^{(i)} P^n)_j = p_j.$$

$\square$

Extra: the Internet search engine Google.

The popular Internet search engine Google, which quickly proved out to be much faster and efficient that all the previously available search engines, is based on a mathematical algorithm which exploits Markov chains to rank webpages. We will briefly try to explain the main ideas behind the algorithm used by Google[14].

Most search engine works in the following way:

- *Crawlers*, that are robot computer programs, search webpages for keywords;
- The information gathered by the crawlers is used to create a *direct index*, that associate each webpage to a list of keywords;
- The direct index is used to create an *inverted index*, which associate to each keyword the relevant webpages;
- When a user enters a search, the inverted index is used to produce a *search output*;
- The search output is *ranked* according to a *page ranking algorithm*;
- The ranked output is seen by the user.

The step which characterizes google is the *page ranking algorithm.* Since the webpages associated to a search are probably thousands and a user will probably be able to see at most the first few 20-30 entries, it is very important that the page-ranking algorithm does produce results which are relevant to the search.

The key idea of Brin and Page used in the *Google page ranking algorithm* is to rank the webpages according to *popularity*, exploiting the Ergodic Theorem for Markov chains as follows. The whole web can be schematize by a huge graph $\mathscr{G}$, where vertices are webpages and arrows from the vertex v_i to the vertex v_j correspond to hyperlinks pointing from the webpage i to the webpage j. The number N of vertices is huge, of the order of 2 million webpages. Let us construct a transition matrix A such that $A_{ij} = 1$ if and only if there is a link between the webpage i and the webpage j. Thus, the graph $\mathscr{G} = \mathscr{G}_A$ coincides with the graph of the subshift space Σ_A defined by the matrix A. Let $N(i)$ be the *number of arrows* exiting the vertex i and pointing to a $j \neq i$ (that is, the number of non self-referential links from the webpage i).

The matrix A is not necessarily *aperiodic.* In order to be able to apply our mathematical tools, let us modify the graph $\mathscr{G}$ as follows. Let us add a vertex v_0 which is connected by an arrow to *all* other vertices. The new transition matrix on $N + 1$ vertices, that we will still call A, is an $(N + 1) \times (N + 1)$ matrix and by construction we have that $A_{0i} = 1$ for all $1 \leq i \leq N$.

Exercise 3.13.2. Verify that the new transition matrix A defined above is *aperiodic.*

Choose a *damping parameter* $p \in (0, 1)$ (effective choices of this parameter have been object of simulations; the parameter currently used by Google is around $p = 0.75$). Use p to define the following probabilities of a stochastic matrix P compatible with A:

$$\begin{array}{lll}
P_{0i} = & \frac{1}{N}, & \text{for all } \ 1 \leq i \leq N; \\
P_{ii} = & 0, & \text{for all } \ 1 \leq i \leq N; \\
P_{i0} = & \begin{cases} 1 & \text{if } \ N(i) = 1, \\ 1-p & \text{if } \ N(i) \neq 1; \end{cases} & \text{for all } \ 1 \leq i \leq N; \\
P_{ij} = & \begin{cases} 0 & \text{if } \ A_{ij} = 0, \\ \frac{p}{N(i)} & \text{if } \ A_{ij} = 1; \end{cases} & \text{for all } \ 1 \leq i, j \leq N; \quad i \neq j.
\end{array}$$

Exercise 3.13.3. Verify that the $(N + 1) \times (N + 1)$ matrix P with entries P_{ij} as above is stochastic and is compatible with the transition matrix A.

[14] As a curiosity, the Google algorithm was created in the mid Ninties by Larry Page and Sergey Brin, that at that time were graduate students at Stanford. The father of Sergey Brin, Misha Brin, is a Professor at University of Maryland working in Dyanamical Systems and is the author of one of the textbooks recommended for this course.

One can understand such a definition as follows. Imagine that an Internet user surfs the web and, after looking at the webpage i, with probability p clicks at random to one of the hyperlinks which appear in the webpage i and points to a webpage j. In addition, once in a while, with probability $1-p$, the user decides not to follow the thread of links anymore and opens a new webpage at random. This process is simulated by the above probabilities: from any webpage i, with probability p one jumps at random to one of the $N(i)$ pages linked by the webpage i, so that each is reached with probability $p/N(i)$. Moreover, one has probability $1-p$ of going to the page 0 and from there equal probability $1/N$ to open any of the other webpages. The idea of the ranking algorithm is that this random simulation can be used to determine which pages are very popular.

By the Perron Frobenius theorem, since P is stochastic and aperiodic, it has a unique probability left eigenvector $\underline{p}$. Google interprets the i^{th} component $0 \leq p_i \leq 1$ of $\underline{p}$ as the popularity *of the webpage i and ranks the webpges in decreasing order of p_i.*

Remark that $\underline{p}$ is the left eigenvector of a matrix with $N = 2$ billions of components. There are computer programs that compute eigenvectors of matrices, but they are not feasible with such a large matrix! *How to compute $\underline{p}$ efficiently?* This time, it is the Ergodic Theorem for Markov chains that comes into play. Take an initial positive vector, for example $q_j = \frac{1}{N}$. If we compute $\underline{q}P^n$, the theorem states that this converges exponentially fast to $\underline{p}$. The page ranking vector $\underline{p}$ can hence be accurately approximated computing iterations of $\underline{q}P^n$.

The Ergodic Theorem for Markov chains also intuitively explains why the vector $\underline{p}$ can be interpreted as vector of polularity from the previously described model for a random Internet surfer. If the surfer starts at random on an initial webpage i (with probability $q_i = \frac{1}{N}$), and jumps to another webpage with probabilities given by P, which describes the above model, p_i represents the limiting probability that, after many clicks, he/she will end up in the webpage i, so p_i is a measure of the *popularity* of the webpage i according to this random simulation of the web and a surfer.